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第三章 热传导方程

$$u_t - \Delta u = f(x,t) \quad x \in \Omega,\ t>0$$

$u(x,0) = \varphi(x)$

(Dirichlet) $u\big|_{\partial\Omega} = g(x,t)$

(Newmann) $\frac{\partial u}{\partial n}\big|_{\partial\Omega} = g(x,t)$

(Robin) $\frac{\partial u}{\partial n} + \sigma u\big|_{\partial\Omega} = g(x,t)$

u表示温度 etc，描述传热过程、扩散过程；

f表示热源

§3.1 初值问题

解法1 分离变量法

$\Omega = [0,l]$、矩形区域、圆盘 etc.

解法2. Fourier 变换法

考察方程 $\begin{cases} u_t - \Delta u = f(x,t) & x \in R^n,\ t>0 \\ u(x,0) = \varphi(x) \end{cases}$

Recall、(R^n上的 Fourier 变换)

$f \in L^1(R^n)$，$\left(\int_{R^n} |f(x)|\,dx < +\infty\right)$

def. $\hat{f}(\xi) = \int_{R^n} f(x) e^{-2\pi i x\cdot\xi}\,dx$

性质 $f \in S(R^n)$，(f光滑，$\forall$阶导数衰减任意快)

则不考虑分部积分边界项

1、令 $(T_{x_0}f)(x) = f(x-x_0)$，则 $\widehat{T_{x_0}f}(\xi) = e^{-2\pi i x_0\cdot\xi}\hat{f}(\xi)$

pr. $\widehat{\tau_{x_0}f}(\xi) = \int_{\mathbb{R}^n} f(x-x_0)\, e^{-2\pi i x\cdot\xi}\, dx$

$= \int_{\mathbb{R}^n} f(y)\, e^{-2\pi i (x_0+y)\cdot\xi}\, dy$

$= e^{-2\pi i x_0\cdot\xi} \int_{\mathbb{R}^n} f(y)\, e^{-2\pi i y\cdot\xi}\, dy$

$= e^{-2\pi i x_0\cdot\xi}\, \hat{f}(\xi)$

2. 令 $(S_\lambda f)(x) = f(\lambda x)$，则 $\widehat{S_\lambda f}(\xi) = \lambda^{-n}\hat{f}(\lambda^{-1}\xi)$

pr. $\widehat{S_\lambda f}(\xi) = \int_{\mathbb{R}^n} f(\lambda x)\, e^{-2\pi i x\cdot\xi}\, dx$

$= \int_{\mathbb{R}^n} f(y)\, e^{-2\pi i \frac{y}{\lambda}\cdot\xi}\, \lambda^{-n} dy$

$= \lambda^{-n} \int_{\mathbb{R}^n} f(y)\, e^{-2\pi i y\cdot\frac{\xi}{\lambda}}\, dy$

$= \lambda^{-n} \int_{\mathbb{R}^n} \hat{f}(\lambda^{-1}\xi)$

3. 对多重指标 $\alpha = (\alpha_1, \cdots, \alpha_n)$，记 $|\alpha| = \alpha_1 + \cdots + \alpha_n$，$x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$

$\partial^\alpha = \partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$，则 $\widehat{\partial^\alpha f}(\xi) = (2\pi i\xi)^\alpha \hat{f}(\xi)$

pr. $\widehat{\partial x_j f}(\xi) = \int_{\mathbb{R}^n} f_{x_j}(x)\, e^{-2\pi i x\cdot\xi}\, dx$

$= -\int_{\mathbb{R}^n} f(x)\, e^{-2\pi i x\cdot\xi}(-2\pi i \xi_j)\, dx$

$= 2\pi i \xi_j \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i x\cdot\xi}\, dx$

$= 2\pi i \xi_j\, \hat{f}(\xi)$

4. $\widehat{(-2\pi i x)^\alpha f}(\xi) = \partial_\xi^\alpha \hat{f}(\xi)$

pr. $\widehat{-2\pi i x_j f}(\xi) = \int_{\mathbb{R}^n} -2\pi i x_j f(x)\, e^{-2\pi i\cdot\xi}\, dx$

$= \int_{\mathbb{R}^n} (e^{-2\pi i\cdot\xi})_{\xi_j} f(x)\, dx$

$= \partial_{\xi_j} \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i\cdot\xi}\, dx$

$= \partial_{\xi_j} \hat{f}(\xi)$

5. 令 $(f*g)(x)=\int_{\mathbb{R}^n} f(x-y)g(y)\,dy=\int_{\mathbb{R}^n} f(y)g(x-y)\,dy$

$\forall f,g\in \mathcal{S}(\mathbb{R}^n)$，则 $\widehat{f*g}(\xi)=\hat f(\xi)\,\hat g(\xi)$

6、逆变换

$\check f(x)=\int_{\mathbb{R}^n} f(\xi)e^{2\pi i x\cdot\xi}dx$，则若 $f\in\mathcal{S}(\mathbb{R}^n)$，有

$\hat f\in \mathcal{S}(\mathbb{R}^n)$，$\check{\hat f}=f$

例. 若 $f(x)=e^{-x^2}$，$x\in\mathbb{R}$，则 $\hat f(\xi)=\sqrt{\pi}\,e^{-\pi^2\xi^2}$

令 $F(\xi)=\hat f(\xi)=\int_{\mathbb{R}} e^{-x^2}e^{-2\pi i x\xi}dx$

$F'(\xi)=\int_{\mathbb{R}} e^{-x^2}(-2\pi i x)e^{-2\pi i x\xi}dx$

$=\pi i\int_{\mathbb{R}} (e^{-x^2})'e^{-2\pi i x\xi}dx$

$=\pi i\,\widehat{f'}(\xi)=\pi i\cdot 2\pi i\xi\,\hat f(\xi)$

$=-2\pi^2\xi F(\xi)$

$F(0)=\int_{\mathbb{R}} e^{-x^2}dx=\sqrt\pi$

$\Rightarrow F(\xi)=\sqrt\pi\, e^{\int -2\pi^2\xi d\xi}=\sqrt\pi\, e^{-\pi^2\xi^2}$

注: $f(x)\in\mathcal S(\mathbb R)$，则 $(e^{-\pi^2\xi^2})^\vee(x)=\frac{1}{\sqrt\pi}e^{-x^2}$

$(e^{-4\pi^2\xi^2 t})^\vee(x)=\frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}$

$n\ge1$时，$(e^{-4\pi^2|\xi|^2t})^\vee(x)=\prod_{j=1}^n(e^{-4\pi^2\xi_j^2t})^\vee(x_j)=\prod_{j=1}^n\frac{1}{2\sqrt{\pi t}}e^{-\frac{x_j^2}{4t}}=\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}$

回到热传导方程的解.

对方程 $\begin{cases}u_t-\Delta u=0 & x\in \mathbb{R}^n,\ t>0\\ u|_{t=0}=\varphi(x) & x\in \mathbb{R}^n\end{cases}$

两边关于x作 Fourier 变换，则 $\begin{cases}\partial_t\hat u(\xi)+4\pi^2|\xi|^2\hat u(\xi)=0\\ \hat u(\xi)|_{t=0}=\hat\varphi(\xi)\end{cases}$

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$$\Rightarrow \hat{u}(\xi,t) = \hat{\varphi}(\xi) e^{-4\pi^2|\xi|^2 t}$$

取逆变换，由性质5 $f*g = (\hat{f}\hat{g})^\vee = \hat{f}^\vee * \hat{g}^\vee$（相乘的逆变换 = 逆变换的卷积）

$$\Rightarrow u(x,t) = (e^{-4\pi^2|\xi|^2 t})^\vee * \varphi$$

$$= \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} * \varphi$$

$$= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} \varphi(y)\, dy$$

貌似在 $t=0$ 有奇性，下说明 $t=0$ 时满足初值

令 $k(x) = \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4}}$ （Heat kernel）

$k_t(x) = t^{-\frac{n}{2}} k(t^{-\frac{1}{2}}x)$, $t>0$

则 $u(x,t) = \int_{\mathbb{R}^n} k_t(x-y)\varphi(y)\,dy = \int_{\mathbb{R}^n} k_t(y)\varphi(x-y)\,dy$

$\{k_t\}_{t>0}$ 有：(i) $\int_{\mathbb{R}^n} k_t(x)\,dx = \int_{\mathbb{R}^n} k(x)\,dx = 1$

左 $= \int_{\mathbb{R}^n} t^{-\frac{n}{2}} k(t^{-\frac{1}{2}}x)\,dx = \int_{\mathbb{R}^n} k(y)\,dy = \frac{1}{(4\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4}}\,dx = 1$

$(\int_{\mathbb{R}} e^{-x^2}\,dx = \sqrt{\pi})$

(ii) $\int_{\mathbb{R}^n} |k_t(x)|\,dx = 1$

(iii) $\forall \eta>0$, $\int_{|x|>\eta} k_t(x)\,dx = \int_{|y|>\frac{\eta}{t^{\frac{1}{2}}}} k(y)\,dy \to 0 \ (t\to 0^+)$

$\{k_t\}_{t>0}$ 为一族逼近恒等

$u(x,t) = k_t * \varphi \xrightarrow{t\to 0^+} \varphi$　　$\varphi \in C(\mathbb{R}^n)$，有界

$$|u(x,t)-\varphi(x)| = \left|\int_{\mathbb{R}^n} k_t(y)\varphi(x-y)\,dy - \int_{\mathbb{R}^n} k_t(y)\varphi(x)\,dy\right|$$

$$= \left|\int_{\mathbb{R}^n} k_t(y)\,(\varphi(x-y)-\varphi(x))\,dy\right|$$

$$\leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{4t}} |\varphi(x-y)-\varphi(x)|\,dy$$

$$\underset{z=\frac{y}{\sqrt{t}}}{=} \frac{1}{(4\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|z|^2}{4}} |\varphi(x-z\sqrt{t})-\varphi(x)|\,dz$$

$$①=\frac{1}{(4\pi)^{\frac{n}{2}}}\int_{|z|\geq R}e^{-\frac{|z|^2}{4}}|\varphi(x-\sqrt{t}z)-\varphi(x)|dz \leq \frac{1}{(4\pi)^{\frac{n}{2}}}\int_{|z|\geq R}e^{-\frac{|z|^2}{4}}dz\cdot 2\|\varphi\|_{L^\infty}\to 0\ (R充分大)$$

$$②=\frac{1}{(4\pi)^{\frac{n}{2}}}\int_{|z|<R}e^{-\frac{|z|^2}{4}}|\varphi(x-\sqrt{t}z)-\varphi(x)|dz$$

由于 φ 连续，有 $\forall\ |z|<R$，$|\varphi(x-z\sqrt{t})-\varphi(x)|<\varepsilon$，$t\to 0^+$

则 ② $\leq C\varepsilon \Rightarrow |u(x,t)-\varphi(x)|\to 0$，$t\to 0^+$

Rmk. 1. 由 K 的光滑性以及 $u(x,t)=K_t(x)*\varphi$ 表明 u 具有光滑性

2. $\sup\limits_x|u(x,t)|\leq\sup\limits_x|\varphi(x)|$

由于 $u(x,t)=\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{-\frac{|y|^2}{4t}}\varphi(x-y)dy$

$$\sup_x|u(x,t)|\leq\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{-\frac{|y|^2}{4t}}\sup_x|\varphi(x)|dy=\sup_x|\varphi(x)|$$

即温度最大值小于初始最大值

3. 热方程沿时间不反向演化，即从末态无法推及初态

4. 无限传播速度，$\forall$ 位置 x 的 $u(x,t)>0$

考虑非齐次方程

对方程 $\begin{cases}u_t-\Delta u=f(x,t) & x\in\mathbb{R}^n,\ t>0\\ u(x,0)=\varphi(x) & x\in\mathbb{R}^n\end{cases}$

取 Fourier 变换，有 $\begin{cases}\partial_t\hat{u}+4\pi^2|\xi|^2\hat{u}=\hat{f}(\xi,t)\\ \hat{u}(\xi,0)=\hat{\varphi}(\xi)\end{cases}$

$$\Rightarrow\hat{u}(\xi,t)=\hat{\varphi}(\xi)e^{-4\pi^2|\xi|^2t}+\int_0^t e^{-4\pi^2|\xi|^2(t-s)}\hat{f}(\xi,s)ds$$

取 Fourier 逆变换，有

$$u(x,t)=\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{-\frac{|x-y|^2}{4t}}\varphi(y)dy+\int_0^t\frac{1}{(4\pi(t-s))^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{-\frac{|x-y|^2}{4(t-s)}}f(y,s)dyds$$

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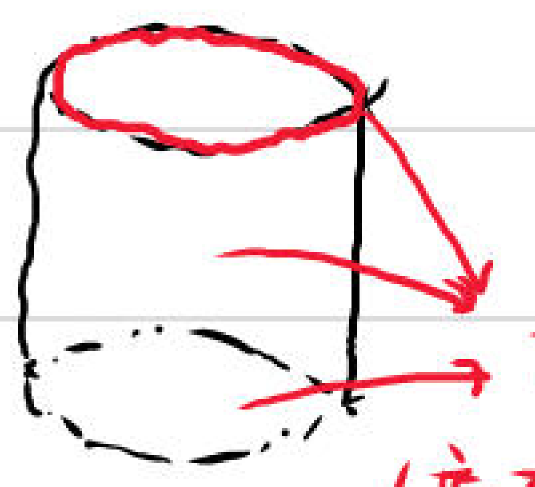
§3.2 极值原理和最大模估计

考虑 $\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega \times \{t>0\} \\ u|_{t=0} = \varphi(x) & \text{in } \Omega \\ u(x,t) = h(x,t) & \text{on } \partial\Omega \times \{t>0\} \end{cases}$



令 $Q_T = \Omega \times (0,T]$，定义抛物边界 $\Gamma_T = \overline{Q_T} \setminus Q_T$

thm1. 极值原理

$u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ 满足 $Lu \triangleq u_t - u_{xx} = f \le 0$

则 $u(x,t)$ 在 $\overline{Q_T}$ 上最大值必在 ∂Q_T 上达到，即

$$\max_{\overline{Q_T}} u(x,t) = \max_{\partial Q_T} u(x,t)$$

pr: 令 $M = \max_{\overline{Q_T}} u(x,t)$，$m = \max_{\partial Q_T} u(x,t)$

step 1. $f<0$，若 $M>m$，

则 f 在 $(x_0,t_0) \in Q_T$ 上达到最大值

则 $u_{xx}(x_0,t_0) \le 0$，$u_t(x_0,t_0) \ge 0$

$Lu(x_0,t_0) = u_t(x_0,t_0) - u_{xx}(x_0,t_0) \ge 0$ 矛盾

step 2. $f=0$，令 $v = u - \varepsilon t$，则 $Lv = Lu - \varepsilon = f - \varepsilon < 0$

由 step 1.

$$\max_{\overline{Q_T}} u - \varepsilon T \le \max_{\overline{Q_T}} v = \max_{\partial Q_T} v \le \max_{\partial Q_T} u$$

由 ε 的任意性，$\max_{\overline{Q_T}} u = \max_{\partial Q_T} u$

Rmk. 则 $Lu = f \ge 0$，则 u 在 $\overline{Q_T}$ 上最小值必在边界处取到

thm 2. 比较原理

$u, v \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ 满足 $Lu \le Lv$，$u|_{\partial Q_T} \le v|_{\partial Q_T}$，则在 $\overline{Q_T}$，$u(x,t) \le v(x,t)$

pr：令 $w(x,t)=u(x,t)-v(x,t)$

$\mathcal{L}w=\mathcal{L}u-\mathcal{L}v\le 0$，$w|_{\partial Q_T}\le 0$

故由极值原理，$\max\limits_{\overline{Q}_T} w=\max\limits_{\partial Q_T} w$

则 $u(x,t)\le v(x,t)\quad \forall (x,t)\in Q_T$

thm 3. 最大模估计

设 $u\in C^{2,1}(Q_T)\cap C(\overline{Q}_T)$ 为方程 $\begin{cases}\mathcal{L}u=f, & x\in(0,l)\times\{t>0\}\\ u|_{t=0}=\varphi(x) & x\in[0,l]\\ u|_{x=0}=g_1(t) & u|_{x=l}=g_2(t)\end{cases}$ 的解，则

$\max\limits_{\overline{Q}_T}|u(x,t)|\le FT+B$

$F=\max\limits_{Q_T}|f|$，$B=\max\left\{\max\limits_{x\in[0,l]}|\varphi(x)|,\ \max\limits_{[0,T]}|g_1(t)|,\ \max\limits_{[0,T]}|g_2(t)|\right\}$

pr：　$v=Ft+B-u$

$\begin{cases}\mathcal{L}v=F-f\ge 0\\ v|_{\partial Q_T}=Ft+B-g_1/g_2\ge 0\end{cases}$

由极大值原理，$\min\limits_{\overline{Q}_T} v=\min\limits_{\partial Q_T} v\ge 0$

$\Rightarrow v(x,t)\ge 0\quad \forall (x,t)\in\overline{Q}_T$

则　$u(x,t)\le Ft+B$

对 $v=Ft+B+u$ 完成相同过程 $\Rightarrow -u(x,t)\le Ft+B$

$\Rightarrow |u(x,t)|\le Ft+B\le FT+B$

Rmk. 可以证明热方程解的唯一性、稳定性

下考虑其余两类边值问题解的唯一性与稳定性

第三类边值问题

$$\begin{cases} u_t - u_{xx} = f(x,t) \\ u(0,t) = \mu_1(t) \quad u_x + hu\,(l,t) = \mu_2(t) \quad h>0 \\ u(x,0) = \varphi(x) \end{cases}$$

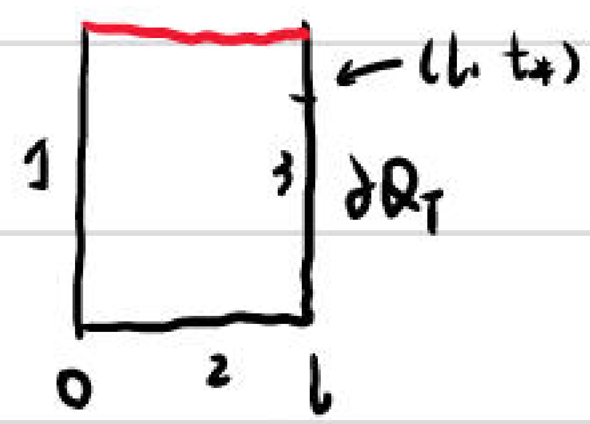
pr: 只需证明 $\begin{cases} u_t - u_{xx} = 0 \\ u(0,t)=0 \qquad u_x + hu\,(l,t) = 0 \qquad h>0 \\ u(x,0)=0 \end{cases}$

只有零解.

否则在 Q_T 上有非零解，其有正的最大值或负的最小值.

$\mathcal{L}u=0$. 由极值原理，最大值最小值均在边界处取到

不妨设正的最大值在边界取到



$u(0,t) = u(x,0) = 0 \Rightarrow$ 正的最大值在3取到

设 u 在 (l,t_*) 取到正的最大值

$u_x(l,t_*) \geq 0$，故 $u_x + hu\,(l,t_*) > 0$ 与边条件矛盾

进而给出了第三类边值下解的唯一性

第二类边值问题

$$\begin{cases} u_t - u_{xx} = f(x,t) \\ u(0,t) = \mu_1(t), \quad u_x(l,t) = \mu_2(t) \\ u(x,0) = \varphi(x) \end{cases}$$

只需证明 $\begin{cases} u_t - u_{xx} = 0 \\ u(0,t)=0 \qquad u_x(l,t)=0 \\ u(x,0)=0 \end{cases}$

只有零解.

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pr: idea $(uw)_x = u_x w + u w_x$

将第二类边值转化为第三类边值

令 $\tilde{u}(x,t) = w(x)u(x,t)$，$u = \frac{\tilde{u}}{w}$

则 $u_t = \frac{\tilde{u}_t}{w}$，$u_x = \frac{\tilde{u}_x w - w_x \tilde{u}}{w^2}$，$u_{xx} = (-\frac{w_{xx}}{w^2} + 2\frac{w_x^2}{w^3})\tilde{u} - 2\frac{w_x}{w^2}\tilde{u}_x + \frac{1}{w}\tilde{u}_{xx}$

$\Rightarrow \frac{\tilde{u}_t}{w} + (\frac{w_{xx}}{w^2} - 2\frac{w_x^2}{w^3})\tilde{u} + \frac{2w_x}{w^2}\tilde{u}_x - \frac{1}{w}\tilde{u}_{xx}$

$$\Rightarrow \begin{cases} \tilde{u}_t - \tilde{u}_{xx} = -2\frac{w_x}{w}\tilde{u}_x - (\frac{w_{xx}}{w} - 2\frac{w_x^2}{w^2})\tilde{u} \\ \tilde{u}(0,t) = 0, \quad \tilde{u}_x(l,t) = w_x u + u_x w\ (l,t) = w_x(l)u(l,t) = w_x(l)\cdot\frac{\tilde{u}(l,t)}{w(l)} \\ \tilde{u}(x,0) = 0 \end{cases}$$

hope $\frac{w_x(l)}{w(l)} = -1$，取 $w(x) = -x+l+1$

$$\text{则} \begin{cases} \tilde{u}_t - \tilde{u}_{xx} = \frac{2}{-x+l+1}\tilde{u}_x + 2\frac{1}{(-x+l+1)^2}\tilde{u} \quad (※) \\ \tilde{u}(0,t) = 0, \quad (\tilde{u}_x + \tilde{u})(l,t) = 0 \\ \tilde{u}(x,0) = 0 \end{cases}$$

令 $v(x,t) = e^{-\lambda t}\tilde{u}(x,t)$，$\tilde{u} = e^{\lambda t}v$

$\tilde{u}_t = v_t e^{\lambda t} + \lambda e^{\lambda t} v$

$\tilde{u}_x = e^{\lambda t} v_x$　$\tilde{u}_{xx} = e^{\lambda t} v_{xx}$

$$\text{则} \begin{cases} v_t - v_{xx} - \frac{2}{-x+l+1}v_x + (\lambda - \frac{2}{(-x+l+1)^2})v = 0 \\ v(0,t) = 0, \quad (v_x + v)(l,t) = 0 \\ v(x,0) = 0 \end{cases}$$

取 $\lambda > 2$，则 $\lambda - \frac{2}{(-x+l+1)^2} > 0$

claim: v 在 Q_T 上 $\bar{Q}$ 的最大值在边界取到

事实上，若在 $v(x_0,t_0) = M > 0$，$(x_0,t_0) \in Q_T$

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则 $V_t(x_0, t_0) \geq 0$，$V_x(x_0, t_0) = 0$，$V_{xx}(x_0, t_0) \leq 0$

与方程条件矛盾

再利用第三类问题的处理给出第三类问题解的唯一性

§3.3 初值问题的最大模估计

考虑实值函数的初值问题

$$\begin{cases} u_t - u_{xx} = f(x,t) & x\in\mathbb{R},\ 0<t\le T \\ u(x,0) = \varphi(x) & x\in\mathbb{R} \end{cases}$$

thm 1. 最大模估计

假设 $u\in C^{2,1}(Q_T)\cap C(\overline{Q_T})$ 为上述问题的有界解

则 $\sup\limits_{Q_T}|u(x,t)| \le T\sup\limits_{Q_T}|f(x,t)| + \sup\limits_{x\in\mathbb{R}}|\varphi(x)|$

令 $F=\sup\limits_{Q_T}|f(x,t)|$，$\Phi=\sup\limits_{x\in\mathbb{R}}|\varphi(x)|$，$M=\sup\limits_{Q_T}|u(x,t)|$

pr: $\forall L>0$，在 $Q_T^L=(-L,L)\times(0,T]$ 上考虑辅助函数

$$w(x,t) = Ft+\Phi+V_L(x,t)\pm u(x,t)$$

$$V_L(x,t) = \frac{M}{L^2}(x^2+2t)$$

(idea: $V_L(x,t)$ 为自由方程 $V_t-V_{xx}=0$ 的解)

则 $w_t - w_{xx} = F\pm f \ge 0$

由极值原理，最小值在边界处取到

$$w\big|_{t=0} = \Phi+\frac{M}{L^2}x^2\pm\phi(x)\ge 0$$

$$w\big|_{x=\pm L} = Ft+\Phi+M\pm u\ge 0$$

$\Rightarrow w(x,t)\ge 0 \quad (x,t)\in Q_T^L$

$\forall (x_0,t_0)\in Q_T$，取 L 充分大，s.t. $(x_0,t_0)\in Q_T^L$

由于 $w(x_0,t_0)\ge 0$，故

$$\begin{aligned}|u(x_0,t_0)| &\le Ft_0+\Phi+\frac{M}{L^2}(x_0^2+2t_0) \\ &\le FT+\Phi+\frac{M}{L^2}(x_0^2+2t_0)\end{aligned}$$

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令 $l\to+\infty$. 则 $|u(x_0,t_0)|\le FT+\Phi$

由 (x_0,t_0) 的任意性, $\sup\limits_{Q_T}|u(x,t)|\le T\sup\limits_{Q_T}|f(x,t)|+\sup\limits_{x\in\mathbb{R}}|\varphi(x)|$

Rmk. 有界性可以放宽为 $|u(x,t)|\le Me^{A|x|^2}$, $(x,t)\in Q_T$

下面用能量估计证明解的唯一性

考虑方程

$$\begin{cases} u_t-u_{xx}=f & (x,t)\in Q_T=(0,l)\times(0,T] \\ u|_{t=0}=\varphi(x) \\ u|_{x=0}=u|_{x=l}=0 \end{cases}$$

thm 2. 能量不等式

设 $u\in C^{1,0}(\overline{Q}_T)\cap C^{2,1}(Q_T)$ 为上述问题的解,

则 $\sup\limits_{0\le t\le T}\int_0^l u^2(x,t)dx+2\int_0^T\int_0^l u_x^2(x,t)\,dxdt$

$\le M\left(\int_0^l \varphi^2(x)\,dx+\int_0^T\int_0^l f^2(x,t)\,dxdt\right)$

pr: $(u_t-u_{xx})u=fu$

左 $=\frac{1}{2}(u^2)_t-(uu_x)_x+(u_x)^2=fu$

在 $[0,l]$ 上积分, 有

$$\frac{d}{dt}\frac{1}{2}\int_0^l u^2dx-uu_x\Big|_{x=0}^{x=l}+\int_0^l u_x^2dx=\int_0^l fu\,dx$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\int_0^l u^2dx\right)+\int_0^l u_x^2dx\le\frac{1}{2}\int_0^l f^2dx+\frac{1}{2}\int_0^l u^2dx$$

$$\Rightarrow \frac{d}{dt}\left(\int_0^l u^2dx\right)\le\int_0^l f^2dx+\int_0^l u^2dx$$

$$\frac{d}{dt}\left(e^{-t}\int_0^l u^2dx\right)\le e^{-t}\int_0^l f^2dx$$

从 $(0,t)$ 积分 $\Rightarrow e^{-t}\int_0^l u^2dx-y(0)\le\int_0^t e^{-s}\int_0^l f^2(x,s)\,dxds$

def $y(t)=\int_0^l u^2(x,t)dx$

则 $y(t) \le e^t y(0) + e^t \int_0^t \int_0^l f^2(x,s)\,dx\,ds$

$\Rightarrow \int_0^l u^2 dx \le e^t \left(\int_0^l \varphi^2 dx + \int_0^t \int_0^l f^2(x,s)\,dx\,ds \right)$

$\le e^T \left(\int_0^l \varphi^2 dx + \int_0^T \int_0^l f^2(x,s)\,dx\,ds \right) \quad \forall\, 0<t\le T$

故 $\frac{d}{dt}\left(\frac{1}{2}\int_0^l u^2 dx\right) + \int_0^l u_x^2 dx \le \frac{1}{2}\int_0^l f^2 dx + \frac{1}{2} e^t \left(\int_0^l \varphi^2 dx + \int_0^T \int_0^l f^2(x,s)\,dx\,ds \right)$

再从 0 到 t 积分，有

$\frac{1}{2}\int_0^l u^2 dx + \int_0^t \int_0^l u_x^2 dx\,dt \le \frac{1}{2}\int_0^l \varphi^2 dx + \frac{1}{2}\int_0^t \int_0^l f^2 dx\,dt$

$+ \frac{1}{2}(e^t - 1)\left[\int_0^l \varphi^2 dx + \int_0^T \int_0^l f^2(x,s)\,dx\,ds \right]$

$\le \frac{1}{2} e^T \int_0^T \int_0^l f^2(x,s)\,dx\,ds \quad \forall\, 0<t\le T$

$\Rightarrow \sup_{0\le t\le T} \int_0^l u^2(x,t)\,dx + 2\int_0^T \int_0^l u_x^2(x,t)\,dx\,dt$

$\le e^T \left(\int_0^l \varphi^2(x)\,dx + \int_0^T \int_0^l f^2(x,t)\,dx\,dt \right)$

Rmk. 可得到解的唯一性与稳定性.