Solutions of QIP2022 exercise 1

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Question (1)

Describe the no-cloning theorem and prove that if quantum cloning is allowed, then superluminal communication is allowed too.

The no-cloning theorem says it is impossible to create identical copies of an arbitrary unknown quantum state.

Note:

1. It is possible to copy linear-independent states probabilistically.
2. It is possible to copy mutually orthogonal states in principle.
3. It is possible to produce imperfect copies of an arbitrary unknown quantum state.

Suppose that two distant parties, Alice and Bob, share a maximally entangled quantum state

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle).$$

If Alice wants to send bit 0(1) to Bob, she measures her particle at Z(X) basis, collapsing her particle to $|0\rangle$ or $|1\rangle$($|+\rangle$ or $|-\rangle$). Then Bob clones his particle to many copies and measures each copy at Z basis.

Let n_+ denote the number of $+1$ outcomes, n_- denote the number of -1 outcomes. One can define the visibility of Bob's Z measurement as $\mathrm{V} \overset{\text{def}}{=} |\frac{\mathrm{n}_+ - \mathrm{n}_-}{\mathrm{n}_+ + \mathrm{n}_-}|$. If Alice measures at Z basis, $\mathrm{V} \approx 1$, while if Alice measures at X basis, $\mathrm{V} \leq \epsilon$, in which $0 < \epsilon \ll 1$. So Bob can distinguish what basis Alice used, i.e. the bit Alice wants to send, immediately after Alice measures her particle without any classical information.

Question (2)

Prove that non-orthogonal states can't be reliably distinguished.

Proof. Suppose there exists two POVMs E_1 and E_2 such that they can distinguish two non-orthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$ reliably, i.e.

$$\mathrm{Prob}(|\psi_1\rangle, 1) = \langle\psi_1|E_1|\psi_1\rangle = 1,$$

$$\mathrm{Prob}(|\psi_2\rangle, 2) = \langle\psi_2|E_2|\psi_2\rangle = 1,$$

$$\mathrm{Prob}(|\psi_2\rangle, 1) = \langle\psi_2|E_1|\psi_2\rangle = 0,$$

$$\mathrm{Prob}(|\psi_1\rangle, 2) = \langle\psi_1|E_2|\psi_1\rangle = 0,$$

where $\mathrm{Prob}(|\psi_i\rangle, j) = \mathrm{tr}(E_j|\psi_i\rangle\langle\psi_i|) = \langle\psi_i|E_j|\psi_i\rangle$ is the probability of measuring $|\psi_i\rangle$ and getting outcome j.

Then from $\langle\psi_1|\mathrm{E}_2|\psi_1\rangle = 0$, we have,

$$\sqrt{\mathrm{E}_2}|\psi_1\rangle = 0.$$

Now since $|\psi_2\rangle$ is not orthogonal to $|\psi_1\rangle$, it can be decomposed into a non-zero component parallel to $|\psi_1\rangle$ and a component orthogonal to $|\psi_1\rangle$,

$$|\psi_2\rangle = \alpha|\psi_1\rangle + \beta|\phi\rangle, |\alpha|^2 + |\beta|^2 = 1.$$

Then, one can reach a contradiction,

$$\sqrt{\mathrm{E}_2}|\psi_2\rangle = \alpha\sqrt{\mathrm{E}_2}|\psi_2\rangle + \beta\sqrt{\mathrm{E}_2}|\phi\rangle = \beta\sqrt{\mathrm{E}_2}|\phi\rangle$$

$$\Rightarrow \langle\psi_2|\mathrm{E}_2|\psi_2\rangle = |\beta|^2\langle\phi|\mathrm{E}_2|\phi\rangle \leq |\beta|^2\sum_{\mathrm{i}}\langle\phi|\mathrm{E}_{\mathrm{i}}|\phi\rangle \leq |\beta|^2 < 1.$$

Question (3)

Write down the commutation relations and anti-commutation relations for the Pauli matrices. Find the eigenvectors, eigenvalues, and diagonal representations of the Pauli matrices.

$$\sigma_\mathrm{i}\sigma_\mathrm{j} = \mathrm{i}\epsilon_{\mathrm{ijk}}\sigma_\mathrm{k} + \delta_{\mathrm{ij}}\mathrm{I}.$$

$$[\sigma_\mathrm{i}, \sigma_\mathrm{j}] = 2\mathrm{i}\epsilon_{\mathrm{ijk}}\sigma_\mathrm{k}.$$

$$\{\sigma_\mathrm{i}, \sigma_\mathrm{j}\} = 2\delta_{\mathrm{ij}}\mathrm{I}.$$

$$\sigma_3 = |0\rangle\langle 0| - |1\rangle\langle 1|.$$

$$\sigma_1 = |+\rangle\langle +| - |-\rangle\langle -|, |+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}, |-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}.$$

$$\sigma_2 = |\mathrm{R}\rangle\langle \mathrm{R}| - |\mathrm{L}\rangle\langle \mathrm{L}|, |\mathrm{R}\rangle = (|0\rangle + \mathrm{i}|1\rangle)/\sqrt{2}, |\mathrm{L}\rangle = (|0\rangle - \mathrm{i}|1\rangle)/\sqrt{2}.$$

Question (4)

Prove the Cauchy – Schwarz inequality that for any two vectors $|v\rangle$ and $|w\rangle$, $|\langle v|w\rangle|^2 \leq \langle v|v\rangle\langle w|w\rangle$.

The proof can be seen in the "Box 2.1" on the page 68 of "Quantum compution and quantum information" by Nielsen.

Simple proof:
The inner product of $|v\rangle + \lambda|w\rangle$ is

$$(\langle v| + \lambda^*\langle w|)(|v\rangle + \lambda|w\rangle) \geq 0,$$

where λ can be any complex number. One chooses $\lambda = -\langle w|v\rangle/\langle w|w\rangle$, then the above inequality reads

$$\langle v|v\rangle - \frac{2|\langle v|w\rangle|^2}{\langle w|w\rangle} + \frac{|\langle v|w\rangle|^2}{(\langle w|w\rangle)^2}\langle w|w\rangle \geq 0 \Rightarrow \langle v|v\rangle\langle w|w\rangle \geq |\langle v|w\rangle|^2.$$

Question (5)

Let $\vec{\mathrm{v}}$ be any real, three-dimensional unit vector and θ a real number. Prove that

$$\exp(\mathrm{i}\theta\vec{\mathrm{v}}\cdot\vec{\sigma}) = \cos(\theta)\mathrm{I} + \mathrm{i}\sin(\theta)\vec{\mathrm{v}}\cdot\vec{\sigma},$$

where $\vec{\mathrm{v}}\cdot\vec{\sigma} = \sum_{\mathrm{i}=1}^{3}\mathrm{v}_\mathrm{i}\sigma_\mathrm{i}$.

Proof.

$$\begin{aligned}\exp(\mathrm{i}\theta\vec{\mathrm{v}}\cdot\vec{\sigma}) &= \sum_{\mathrm{n}=0}^{\infty}\frac{1}{\mathrm{n}!}(\mathrm{i}\theta\vec{\mathrm{v}}\cdot\vec{\sigma})^{\mathrm{n}} \\ &= \sum_{\mathrm{k}=0}^{\infty}\frac{(-1)^{\mathrm{k}}}{(2\mathrm{k})!}\theta^{2\mathrm{k}}(\vec{\mathrm{v}}\cdot\vec{\sigma})^{2\mathrm{k}} + \sum_{\mathrm{k}=0}^{\infty}\frac{\mathrm{i}(-1)^{\mathrm{k}}}{(2\mathrm{k}+1)!}\theta^{2\mathrm{k}+1}(\vec{\mathrm{v}}\cdot\vec{\sigma})^{2\mathrm{k}+1}.\end{aligned}$$

To simplify, we calculate $(\vec{\mathrm{v}} \cdot \vec{\sigma})^2$,

$$(\vec{\mathrm{v}} \cdot \vec{\sigma})^2 = \mathrm{v_i v_j} \sigma_\mathrm{i} \sigma_\mathrm{j} = \mathrm{v_i v_j}(\delta_\mathrm{ij} \mathrm{I} + \mathrm{i}\epsilon_\mathrm{ijk} \sigma_\mathrm{k})$$
$$= \|\mathrm{v}\|^2 \mathrm{I} + \mathrm{i}(\vec{\mathrm{v}} \times \vec{\mathrm{v}})_\mathrm{k} \sigma_\mathrm{k} = \mathrm{I}.$$

$$\exp(\mathrm{i}\theta \vec{\mathrm{v}} \cdot \vec{\sigma}) = \sum_{\mathrm{k}=0}^{\infty} \frac{(-1)^\mathrm{k}}{(2\mathrm{k})!} \theta^{2\mathrm{k}} \mathrm{I} + \sum_{\mathrm{k}=0}^{\infty} \frac{\mathrm{i}(-1)^\mathrm{k}}{(2\mathrm{k}+1)!} \theta^{2\mathrm{k}+1} (\vec{\mathrm{v}} \cdot \vec{\sigma})$$
$$= \cos(\theta)\mathrm{I} + \mathrm{i}\sin(\theta) \vec{\mathrm{v}} \cdot \vec{\sigma}.$$

Generally, for any 2×2 matrix A satisfying $\mathrm{A}^2 = \mathrm{I}$, we have

$$\exp(\pm \mathrm{i}\theta \mathrm{A}) = \cos(\theta)\mathrm{I} \pm \mathrm{i}\sin(\theta)\mathrm{A}.$$

Note: The rotation $\exp(\mathrm{i}\theta \vec{\mathrm{v}} \cdot \vec{\sigma})$ rotates the Bloch vector about $\vec{\mathrm{v}}$ by angle 2θ. This is related to the SU(2) symmetry of the Bloch sphere.

Question (6)

Prove that for any 2-dimension linear operator A,

$$\mathrm{A} = \frac{1}{2}\mathrm{tr}(\mathrm{A})\mathrm{I} + \frac{1}{2}\sum_{\mathrm{k}=1}^{3}\mathrm{tr}(\mathrm{A}\sigma_{\mathrm{k}})\sigma_{\mathrm{k}},$$

in which $\sigma_{\mathrm{k}}(\mathrm{k}=1,2,3)$ are Pauli matrices.

Note: This equation says that Pauli matrices a complete basis of the linear space of 2-dimensional linear operator,

$$\mathrm{A} = \langle \mathrm{I}, \mathrm{A}\rangle \mathrm{I} + \langle \sigma_1, \mathrm{A}\rangle \sigma_1 + \langle \sigma_2, \mathrm{A}\rangle \sigma_2 + \langle \sigma_3, \mathrm{A}\rangle \sigma_3,$$

where the (normalized) inner product of matrices is called Hilbert-Schmidt inner product:

$$\langle \mathrm{M}, \mathrm{N}\rangle = \frac{\mathrm{tr}(\mathrm{M}^\dagger \mathrm{N})}{2}.$$

Proof. Suppose that $\mathrm{A} = \mathrm{a_j}\sigma_\mathrm{j}$, where $\sigma_0 = \mathrm{I}$ and $\sigma_\mathrm{k}(\mathrm{k} = 1, 2, 3)$ are Pauli matrices, then

$$\mathrm{A}\sigma_\mathrm{k} = \mathrm{a_j}\sigma_\mathrm{j}\sigma_\mathrm{k}.$$

Since

$$\mathrm{tr}(\sigma_\mathrm{j}\sigma_\mathrm{k}) = \mathrm{tr}(\delta_\mathrm{jk}\mathrm{I} + \mathrm{i}\epsilon_\mathrm{jkl}\sigma_\mathrm{l}) = 2\delta_\mathrm{jk},$$

then

$$\mathrm{tr}(\mathrm{A}\sigma_\mathrm{k}) = \sum_{\mathrm{j}=0}^{3} \mathrm{tr}(\mathrm{a_j}\sigma_\mathrm{j}\sigma_\mathrm{k}) = \sum_{\mathrm{j}=0}^{3} 2\mathrm{a_j}\delta_\mathrm{jk} = 2\mathrm{a_k} \Rightarrow \mathrm{a_k} = \frac{1}{2}\mathrm{tr}(\mathrm{A}\sigma_\mathrm{k}),$$

then we get

$$\mathrm{A} = \frac{1}{2}\mathrm{tr}(\mathrm{A})\mathrm{I} + \frac{1}{2}\sum_{\mathrm{k}=1}^{3} \mathrm{tr}(\mathrm{A}\sigma_\mathrm{k})\sigma_\mathrm{k}. \qquad \square$$

Question (7)

Let ρ be a density operator.

(1). Show that ρ can be written as

$$\rho = \frac{\mathrm{I} + \vec{\mathrm{r}} \cdot \vec{\sigma}}{2}$$

where $\vec{\mathrm{r}}$ is a real three-dimensional vector such that $||\vec{\mathrm{r}}|| \leq 1$.

(2). Show that $\mathrm{tr}(\rho^2) \leq 1$, with equality if and only if ρ is a pure state.

(3). Show that a state ρ is a pure state if and only if $||\vec{\mathrm{r}}|| = 1$.

(1). Using the conclusion derived in the previous problem, we get

$$\rho = \frac{1}{2}\mathrm{tr}(\rho)\mathrm{I} + \frac{1}{2}\sum_{\mathrm{i}=1}^{3}\mathrm{tr}(\rho\sigma_{\mathrm{i}})\sigma_{\mathrm{i}}.$$

By defining $\mathrm{r_i} = \mathrm{tr}(\rho\sigma_{\mathrm{i}}), (\mathrm{i} = 1, 2, 3)$, we get

$$\rho = \frac{\mathrm{I} + \vec{\mathrm{r}} \cdot \vec{\sigma}}{2}.$$

(2). Do the spectral decomposition of ρ: $\rho = \sum_{\mathrm{i}} \mathrm{p_i}|\phi_{\mathrm{i}}\rangle\langle\phi_{\mathrm{i}}|$, then

$$\begin{aligned}
\rho^2 &= \sum_{\mathrm{i}} \mathrm{p_i}|\phi_{\mathrm{i}}\rangle\langle\phi_{\mathrm{i}}| \sum_{\mathrm{j}} \mathrm{p_j}|\phi_{\mathrm{j}}\rangle\langle\phi_{\mathrm{j}}| \\
&= \sum_{\mathrm{i,j}} \mathrm{p_i p_j}|\phi_{\mathrm{i}}\rangle\langle\phi_{\mathrm{i}}|\phi_{\mathrm{j}}\rangle\langle\phi_{\mathrm{j}}| \\
&= \sum_{\mathrm{i}} \mathrm{p_i^2}|\phi_{\mathrm{i}}\rangle\langle\phi_{\mathrm{i}}|,
\end{aligned}$$

thus,

$$\begin{aligned}\mathrm{tr}(\rho^2) &= \mathrm{tr}(\sum_{\mathrm{i}} \mathrm{p}_{\mathrm{i}}^2 |\phi_{\mathrm{i}}\rangle\langle\phi_{\mathrm{i}}|) \\ &= \sum_{\mathrm{i}} \mathrm{p}_{\mathrm{i}}^2 \langle\phi_{\mathrm{i}}|\phi_{\mathrm{i}}\rangle = \sum_{\mathrm{i}} \mathrm{p}_{\mathrm{i}}^2.\end{aligned}$$

Since $\sum_{\mathrm{i}} \mathrm{p}_{\mathrm{i}} = 1$, then

$$\mathrm{tr}(\rho^2) = \sum_{\mathrm{i}} \mathrm{p}_{\mathrm{i}}^2 \leq \sum_{\mathrm{i}} \mathrm{p}_{\mathrm{i}} = 1,$$

with equality if and only if

$$\mathrm{p}_{\mathrm{j}} = 1, \mathrm{p}_{\mathrm{i}\neq\mathrm{j}} = 0,$$

which indicates that the state ρ is a pure state.

(3). Similar to Question (5),

$$(\vec{\mathrm{r}} \cdot \vec{\sigma})^2 = \sum_{\mathrm{i}} \mathrm{r}_{\mathrm{i}}^2 \mathrm{I} = \|\vec{\mathrm{r}}\|^2 \mathrm{I},$$

thus,

$$\rho^2 = (\frac{\mathrm{I} + \vec{\mathrm{r}} \cdot \vec{\sigma}}{2})^2 = \frac{\mathrm{I} + 2\vec{\mathrm{r}} \cdot \vec{\sigma} + \|\vec{\mathrm{r}}\|^2 \mathrm{I}}{4}.$$

$$\mathrm{tr}(\rho^2) = \frac{1}{4} \mathrm{tr}(\mathrm{I} + 2\vec{\mathrm{r}} \cdot \vec{\sigma} + ||\vec{\mathrm{r}}||^2 \mathrm{I}) = \frac{1}{2}(1 + ||\vec{\mathrm{r}}||^2).$$

Since

$$\mathrm{tr}(\rho^2) \leq 1,$$

with equality holds if and only if ρ is a pure state, then

$$||\vec{\mathrm{r}}||^2 \leq 1.$$

$||\vec{\mathrm{r}}|| = 1$ if and only if ρ is a pure state.

Question (8)

$\rho_A = \frac{I + \vec{n_A} \cdot \vec{\sigma}}{2}$, $\rho_B = \frac{I + \vec{n_B} \cdot \vec{\sigma}}{2}$, prove that $\mathrm{tr}(\rho_A \rho_B) = \frac{1 + \vec{n_A} \cdot \vec{n_B}}{2}$.

Using

$$\sigma_i \sigma_j = i\epsilon_{ijk}\sigma_k + \delta_{ij} I,$$

one gets

$$(\vec{n_A} \cdot \vec{\sigma})(\vec{n_B} \cdot \vec{\sigma}) = n_{Ai}\sigma_i n_{Bj}\sigma_j = n_{Ai}n_{Bj}(\delta_{ij}I + i\epsilon_{ijk}\sigma_k) = n_{Ai}n_{Bi}I + i\epsilon_{ijk}n_{Ai}n_{Bj}$$

$$\rho_A \rho_B = \frac{1}{4}(I + \vec{n_A} \cdot \vec{\sigma} + \vec{n_B} \cdot \vec{\sigma} + \vec{n_A} \cdot \vec{n_B} I + i\epsilon_{ijk}n_{Ai}n_{Bj}\sigma_k).$$

As $\mathrm{tr}(\sigma_i) = 0, \mathrm{tr}(I) = 2,$

$$\mathrm{tr}(\rho_A \rho_B) = \frac{1 + \vec{n_A} \cdot \vec{n_B}}{2}.$$

Question

Consider an experiment in which we prepare the state $|0\rangle$ with the probability $|\mathrm{C}_0|^2$,and the state $|1\rangle$ with the probability $|\mathrm{C}_1|^2$. How to describe this type of quantum state? Compare the differences and similarities between it with the state $\mathrm{C}_0|0\rangle + \mathrm{C}_1 \mathrm{e}^{\mathrm{i}\theta}|1\rangle$.

This state is a mixed state, whose density matrix is

$$\rho = |\mathrm{C}_0|^2|0\rangle\langle 0| + |\mathrm{C}_1|^2|1\rangle\langle 1| = \begin{pmatrix} |\mathrm{C}_0|^2 & 0 \\ 0 & |\mathrm{C}_1|^2 \end{pmatrix}.$$

The state $|\psi\rangle = \mathrm{C}_0|0\rangle + \mathrm{C}_1 \mathrm{e}^{\mathrm{i}\theta}|1\rangle$ is a pure state, whose density matrix is

$$\rho' = |\mathrm{C}_0|^2|0\rangle\langle 0| + |\mathrm{C}_1|^2|1\rangle\langle 1| + \mathrm{C}_0\mathrm{C}_1\mathrm{e}^{-\mathrm{i}\theta}|0\rangle\langle 1| + \mathrm{C}_0\mathrm{C}_1\mathrm{e}^{\mathrm{i}\theta}|1\rangle\langle 0|$$

$$= \begin{pmatrix} |\mathrm{C}_0|^2 & \mathrm{C}_0\mathrm{C}_1\mathrm{e}^{-\mathrm{i}\theta} \\ \mathrm{C}_0\mathrm{C}_1\mathrm{e}^{\mathrm{i}\theta} & |\mathrm{C}_1|^2 \end{pmatrix}.$$

$$\rho = \begin{pmatrix} |C_0|^2 & 0 \\ 0 & |C_1|^2 \end{pmatrix}, \rho' = \begin{pmatrix} |C_0|^2 & C_0 C_1 e^{-i\theta} \\ C_0 C_1 e^{i\theta} & |C_1|^2 \end{pmatrix}.$$

The latter one has nonzero non-diagonal terms, i.e. coherence.

When measuring these two states, if $\{|0\rangle, |1\rangle\}$ basis is used, the probabilities we get $|0\rangle$ and $|1\rangle$ are same; if other basis is used, the probabilities would be different.

Question (10)

Suppose a two particle pure state is of the form $|\Phi\rangle_{\mathrm{AB}} = \frac{1}{\sqrt{2}}|0\rangle(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle) + \frac{1}{\sqrt{2}}|1\rangle(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle)$.
(1)Calculate the reduced density matrices ρ_{A} and ρ_{B}.(2)Do Schmidt decomposition.

(1).

$$\begin{aligned}\rho_{\mathrm{AB}} =&\frac{1}{8}(|00\rangle\langle 00| + \sqrt{3}|00\rangle\langle 01| + \sqrt{3}|00\rangle\langle 10| + |00\rangle\langle 11| \\ &+ \sqrt{3}|01\rangle\langle 00| + 3|01\rangle\langle 01| + 3|01\rangle\langle 10| + \sqrt{3}|01\rangle\langle 11| \\ &+ \sqrt{3}|10\rangle\langle 00| + 3|10\rangle\langle 01| + 3|10\rangle\langle 10| + \sqrt{3}|10\rangle\langle 11| \\ &+ |11\rangle\langle 00| + \sqrt{3}|11\rangle\langle 01| + \sqrt{3}|11\rangle\langle 10| + |11\rangle\langle 11|)\end{aligned}$$

$$\rho_{\mathrm{A}} = \rho_{\mathrm{B}} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| + \frac{\sqrt{3}}{4}|0\rangle\langle 1| + \frac{\sqrt{3}}{4}|1\rangle\langle 0|.$$

(2). The eigenvalues of ρ_{A} and ρ_{B} are same, $\lambda_{1,2} = \frac{1}{4}(2 \pm \sqrt{3})$, and we choose eigenvectors of ρ_{A} and ρ_{B} as:

$$|\lambda_1^{\mathrm{A}}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |\lambda_2^{\mathrm{A}}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

$$|\lambda_1^{\mathrm{B}}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |\lambda_2^{\mathrm{B}}\rangle = -\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Then,

$$\begin{aligned} |\Phi\rangle_{\mathrm{AB}} &= \sqrt{\lambda_1}|\lambda_1\rangle_{\mathrm{A}}|\lambda_1\rangle_{\mathrm{B}} + \sqrt{\lambda_2}|\lambda_2\rangle_{\mathrm{A}}|\lambda_2\rangle_{\mathrm{B}} \\ &= \frac{1+\sqrt{3}}{2\sqrt{2}}|\lambda_1\rangle_{\mathrm{A}}|\lambda_1\rangle_{\mathrm{B}} + \frac{\sqrt{3}-1}{2\sqrt{2}}|\lambda_2\rangle_{\mathrm{A}}|\lambda_2\rangle_{\mathrm{B}} \end{aligned}$$

Question (11)

Prove that suppose $|\psi\rangle$ is a pure state of a composite system, AB. Then there exist orthonormal states $|\mathrm{i_A}\rangle$ for system A, and orthonormal states $|\mathrm{i_B}\rangle$ for system B such that

$$|\psi\rangle = \sum_{\mathrm{i}} \lambda_{\mathrm{i}} |\mathrm{i_A}\rangle|\mathrm{i_B}\rangle,$$

where λ_{i} are non-negative real numbers satisfying $\sum_{\mathrm{i}} \lambda_{\mathrm{i}}^2 = 1$ known as Schmidt coefficients.

The proof can be seen in the "Theorem 2.7" on the page 109 of "Quantum computation and quantum information" by Nielsen.

Similarly, for the case where the state spaces of A and B have different dimension, we can always write $|\psi\rangle$ as

$$|\psi\rangle = \sum_{j=1}^{n}\sum_{k=1}^{m} a_{jk}|j\rangle|k\rangle,$$

where $\{|j\rangle\}$ and $\{|k\rangle\}$ are orthonormal bases. Without loss of generality, we suppose $n > m$. Then by the singular value decomposition, $a = u\begin{pmatrix} d \\ 0 \end{pmatrix}v$, where u is a $n \times n$ unitary matrix, v is a $m \times m$ unitary matrix, d is a $m \times n$ diagonal matrix with non-negative real entries, and $\begin{pmatrix} d \\ 0 \end{pmatrix}$ denotes the $n \times m$ matrix whose $(m+1)$th to nth rows are 0.

Thus, $|\psi\rangle = \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m} u_{ji} d_{ii} v_{ik} |j\rangle|k\rangle$. Let $|i_A\rangle \equiv \sum_{j=1}^{n} u_{ji}|j\rangle$, $|i_B\rangle \equiv \sum_{k=1}^{m} v_{ik}$ and $\lambda_i = d_{ii}$, then

$$|\psi\rangle = \sum_{i=1}^{m} \lambda_i |i_A\rangle|i_B\rangle.$$

$\{|i_A\rangle\}$ and $\{|i_B\rangle\}$ both are orthonormal sets since unitary operators transform one orthonormal basis to another orthonormal basis.

Moreover, since $|\psi\rangle$ is normalized, we have

$$1 = \langle\psi|\psi\rangle = \sum_{i=1}^{m} \lambda_i \langle i_A|\langle i_B| \sum_{j=1}^{m} \lambda_j |j_A\rangle|j_B\rangle = \sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_i\lambda_j\delta_{ij}\delta_{ij} = \sum_{i=1}^{m} \lambda_i^2.$$

Question (12)

Suppose $\{|\psi_i\rangle\}, \{|\tilde{\psi}_i\rangle\}$ are two sets of normalized states in space H and they satisfy the conditions that $\langle\psi_i|\psi_j\rangle = \langle\tilde{\psi}_i|\tilde{\psi}_j\rangle$ for $\forall i, j$, then prove that there exist a transformation U, s.t. $U|\psi_i\rangle = |\tilde{\psi}_i\rangle$, and construct this U transformation.

Using Gram-Schmidt method to first orthogonalize $\{|\psi_i\rangle\}$ and $\left\{|\tilde{\psi}_i\rangle\right\}$ respectively, we have

$$
\begin{aligned}
&|\alpha_1\rangle = |\psi_1\rangle, \\
&|\alpha_2\rangle = |\psi_2\rangle - \langle\alpha_1|\psi_2\rangle|\alpha_1\rangle, \\
&\ldots \\
&|\alpha_i\rangle = |\psi_i\rangle - \sum_{n=1}^{i-1}\langle\alpha_n|\psi_i\rangle|\alpha_n\rangle.
\end{aligned}
$$

Then we normalize them to get $|\beta_i\rangle = |\alpha_i\rangle/\sqrt{\langle\alpha_i|\alpha_i\rangle}$. Similarly, we get orthogonal set $\{|\tilde{\alpha}_i\rangle\}$ and orthonormal set $\left\{|\tilde{\beta}_i\rangle\right\}$. Note that one may need to add additional vectors or remove redundant vectors to make $\{|\beta_i\rangle\}$ and $\left\{|\tilde{\beta}_i\rangle\right\}$ be orthonormal bases of the Hilbert space H.

Now, we construct U as

$$U = \sum_j |\tilde{\beta}_j\rangle\langle\beta_j|.$$

Since $\{|\beta_i\rangle\}$ and $\left\{|\tilde{\beta}_i\rangle\right\}$ are orthonormal, $U = \sum_j |\tilde{\beta}_j\rangle\langle\beta_j|$ is indeed unitary. Then,

$$U|\psi_i\rangle = \sum_j |\tilde{\beta}_j\rangle\langle\beta_j|\psi_i\rangle \stackrel{?}{=} \sum_j |\tilde{\beta}_j\rangle\langle\tilde{\beta}_j|\tilde{\psi}_i\rangle = |\tilde{\psi}_i\rangle.$$

To prove $\langle \beta_{\mathrm{i}}|\psi_{\mathrm{j}}\rangle = \langle \tilde{\beta}_{\mathrm{i}}|\tilde{\psi}_{\mathrm{j}}\rangle, \forall \mathrm{i}, \mathrm{j}$, we use the strong form of mathematical induction to prove $\langle \alpha_{\mathrm{i}}|\psi_{\mathrm{j}}\rangle = \langle \tilde{\alpha}_{\mathrm{i}}|\tilde{\psi}_{\mathrm{j}}\rangle, \forall \mathrm{i}, \mathrm{j}$ first. It is obvious that $\forall \mathrm{j}, \langle \alpha_1|\psi_{\mathrm{j}}\rangle = \langle \tilde{\alpha}_1|\tilde{\psi}_{\mathrm{j}}\rangle$ and $\langle \alpha_2|\psi_{\mathrm{j}}\rangle = \langle \tilde{\alpha}_2|\tilde{\psi}_{\mathrm{j}}\rangle$ from the condition $\langle \psi_{\mathrm{i}}|\psi_{\mathrm{j}}\rangle = \langle \tilde{\psi}_{\mathrm{i}}|\tilde{\psi}_{\mathrm{j}}\rangle$. Then assume that when $\mathrm{i} < \mathrm{k}+1$, the equality $\langle \alpha_{\mathrm{i}}|\psi_{\mathrm{j}}\rangle = \langle \tilde{\alpha}_{\mathrm{i}}|\tilde{\psi}_{\mathrm{j}}\rangle, \forall \mathrm{j}$ holds. Thus when $\mathrm{i} = \mathrm{k}+1$, we have

$$\begin{aligned}
\langle \alpha_{\mathrm{k}+1}|\psi_{\mathrm{j}}\rangle &= \langle \psi_{\mathrm{k}+1}|\psi_{\mathrm{j}}\rangle - \sum_{\mathrm{n}=1}^{\mathrm{k}} \langle \psi_{\mathrm{k}+1}|\alpha_{\mathrm{n}}\rangle\langle \alpha_{\mathrm{n}}|\psi_{\mathrm{j}}\rangle \\
&= \langle \tilde{\psi}_{\mathrm{k}+1}|\tilde{\psi}_{\mathrm{j}}\rangle - \sum_{\mathrm{n}=1}^{\mathrm{k}} \langle \tilde{\psi}_{\mathrm{k}+1}|\tilde{\alpha}_{\mathrm{n}}\rangle\langle \tilde{\alpha}_{\mathrm{n}}|\tilde{\psi}_{\mathrm{j}}\rangle \\
&= \langle \tilde{\alpha}_{\mathrm{k}+1}|\tilde{\psi}_{\mathrm{j}}\rangle.
\end{aligned}$$

Therefore, one can conclude that $\langle \alpha_{\mathrm{i}}|\psi_{\mathrm{j}}\rangle = \langle \tilde{\alpha}_{\mathrm{i}}|\tilde{\psi}_{\mathrm{j}}\rangle, \forall \mathrm{i}, \mathrm{j}$. Similarly, one can prove that $\langle \alpha_{\mathrm{i}}|\alpha_{\mathrm{i}}\rangle = \langle \tilde{\alpha}_{\mathrm{i}}|\tilde{\alpha}_{\mathrm{i}}\rangle, \forall \mathrm{i}$. Thus, $\langle \beta_{\mathrm{i}}|\psi_{\mathrm{j}}\rangle = \langle \tilde{\beta}_{\mathrm{i}}|\tilde{\psi}_{\mathrm{j}}\rangle, \forall \mathrm{i}, \mathrm{j}$.

Question (13)

Suppose ABC is a three component quantum system. Show by example that there are pure quantum states ψ of such systems which can not be written in the form

$$|\psi\rangle = \sum_{\mathrm{i}} \lambda_{\mathrm{i}} |\mathrm{i_A}\rangle|\mathrm{i_B}\rangle|\mathrm{i_C}\rangle$$

where λ_{i} are real numbers, and $|\mathrm{i_A}\rangle, |\mathrm{i_B}\rangle, |\mathrm{i_C}\rangle$ are orthonormal bases of the respective systems.

For example,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |011\rangle).$$

Generally speaking, assume that $|\psi\rangle$ can be written in the form

$$|\psi\rangle = \sum_{\mathrm{i}} \lambda_{\mathrm{i}} |\mathrm{i}_{\mathrm{A}}\rangle |\mathrm{i}_{\mathrm{B}}\rangle |\mathrm{i}_{\mathrm{C}}\rangle,$$

then the reduced density matrices ρ_{A}, ρ_{B} and ρ_{C} shall have the same eigenvalues λ_{i}^2.

However, for $|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |011\rangle)$, ρ_{A}, ρ_{B} and ρ_{C} don't have common eigenvalues. So $|\psi\rangle$ can not be written in the form described as above.