中国科学技术大学 2019–2020学年夏季学期

《基础微积分3》答案

一、填空题.

1.$\underline{\frac{3}{2}}$,　2.$\underline{-\frac{4}{9\pi}}$,　3. $\underline{\text{收敛}}$.　4.$\underline{-\int_y^{y^2} x^2 e^{-x^2 y}\mathrm{d}x + 2ye^{-y^5} - e^{-y^3}}$.　5.$\underline{\{a \neq 0\}}$.

二、1.

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\mathrm{d}x = \frac{1}{\pi}\left(\int_0^{\pi}\pi\mathrm{d}x + \int_\pi^{2\pi}(2\pi - x)\mathrm{d}x\right) = \frac{3}{2}\pi \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\mathrm{d}x = \frac{1}{\pi}\left(\int_0^{\pi}\pi\cos nx\mathrm{d}x + \int_\pi^{2\pi}(2\pi - x)\cos nx\mathrm{d}x\right) = \frac{(-1)^n - 1}{n^2\pi} \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\mathrm{d}x = \frac{1}{\pi}\left(\int_0^{\pi}\pi\sin nx\mathrm{d}x + \int_\pi^{2\pi}(2\pi - x)\sin nx\mathrm{d}x\right) = \frac{1}{n}
\end{aligned}
$$

$$
\begin{aligned}
f(x) &\sim \frac{3\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right) \\
&= \begin{cases} f(x), & x \neq 2n\pi \\ \frac{\pi}{2}, & x = 2n\pi \end{cases}, \qquad n \in \mathbb{Z}
\end{aligned}
$$

2. 当$x = 0$时成立

$$\frac{\pi}{2} = \frac{3\pi}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2\pi} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty}\frac{1}{4n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

由Parseval等式可得

$$\frac{1}{2}\frac{9}{4}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{(2n-1)^4\pi^2} + \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{\pi}\int_0^{\pi}\pi^2\mathrm{d}x + \int_\pi^{2\pi}(2\pi - x)^2\mathrm{d}x = \frac{4}{3}\pi^2$$

所以$\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$.

三、$e^{-|x|}$是偶函数，

1.$\displaystyle F(\lambda) = \int_{-\infty}^{+\infty} f(x)e^{-i\lambda x} = 2\int_0^{+\infty} e^{-x}\cos\lambda x\mathrm{d}x = \frac{2}{1+\lambda^2}$.

2.$\displaystyle F'(\lambda) = -i\int_{-\infty}^{+\infty} xf(x)e^{-i\lambda x}\mathrm{d}x = -i\mathcal{F}[xf(x)]$

所以$G(\lambda) = \mathcal{F}[xe^{-|x|}] = -\frac{1}{i}F'(\lambda) = \frac{-2i\lambda}{(1+\lambda^2)^2}$

3.利用Fourier逆变换

$$f(x) = \frac{1}{\pi}\int_0^{+\infty}\frac{2\cos\lambda x}{1+\lambda^2}\mathrm{d}\lambda = e^{-|x|}$$

可以得到　$\displaystyle\int_0^{+\infty}\frac{\cos\lambda x}{1+\lambda^2}\mathrm{d}\lambda = \frac{\pi}{2}e^{-|x|}$.

四、根据含参量常义积分的性质，

$$J_n'(x)=\frac{1}{\pi}\int_0^{\pi}\sin(nt-x\sin t)\sin t\mathrm{d}t$$
$$J_n''(x)=-\frac{1}{\pi}\int_0^{\pi}\cos(nt-x\sin t)\sin^2 t\mathrm{d}t$$

代入方程左边

$$\begin{aligned}
&x^2J_n''(x)+xJ_n'(x)+(x^2-n^2)J_n(x)\\
=&\frac{1}{\pi}\int_0^{\pi}(-x^2\sin^2 t+x^2-n^2)\cos(nt-x\sin t)\mathrm{d}t+\frac{1}{\pi}\int_0^{\pi}x\sin t\sin(nt-x\sin t)\mathrm{d}t\\
=&\frac{1}{\pi}\int_0^{\pi}-(x\cos t+n)\cos(nt-x\sin t)\mathrm{d}(nt-x\sin t)+\frac{1}{\pi}\int_0^{\pi}x\sin t\sin(nt-x\sin t)\mathrm{d}t\\
=&\frac{1}{\pi}\left[-(x+\cos nt)\sin(nt-x\sin t)\Big|_{t=0}^{\pi}+\int_0^{\pi}-x\sin t\sin(nt-x\sin t)\mathrm{d}t\right]+\frac{1}{\pi}\int_0^{\pi}x\sin t\sin(nt-x\sin t)\mathrm{d}t\\
=&0
\end{aligned}$$

所以$J_n(x)$满足方程$x^2J_n''(x)+xJ_n'(x)+(x^2-n^2)J_n(x)=0.$

五、证明： 当$p>p_0>0$时，$x^{p-1}\ln^2 x<x^{p_0-1}\ln^2 x.$

$\int_0^1 x^{p_0-1}\ln^2 x\mathrm{d}x$，瑕点为$x=0$, $\lim\limits_{t\to 0^+}\frac{x^{p_0-1}\ln^2 x}{x^{p_0/2-1}}=\lim\limits_{t\to 0^+}x^{\frac{p_0}{2}}\ln^2 x=0,$

$\int_0^1 x^{\frac{p_0}{2}-1}\mathrm{d}x$收敛，由比较判别法，$\int_0^1 x^{p_0-1}\ln^2 x\mathrm{d}x$收敛。 由Weierstrass判别法，$\int_0^1 x^{p-1}\ln^2 x\mathrm{d}x$ 在 $p>p_0>0$ 上一致收敛.

六、证明：任取任取$[a,b]\subset(0,+\infty)$，

$\left|\int_1^A\cos x\mathrm{d}x\right|\le 2$，即积分$\int_1^A\cos x\mathrm{d}x$ 关于$\alpha\in[a,b]$ 一致有界；

$\dfrac{1}{x^\alpha}$ 关于x 单调，且$\forall x\in[a,b],\frac{1}{x^\alpha}\le\frac{1}{x^a}$ 成立，所以当$x\in[a,b]$ 时，$\dfrac{1}{x^\alpha}$ 关于α 一致趋于零。

由Dirichlet 判别法，可知$\int_1^{+\infty}\dfrac{\cos x}{x^\alpha}\mathrm{d}x$ 在$[a,b]$ 上一致收敛，从而 在$[a,b]$ 上连续.

对任意$x_0\in(0,+\infty)$, 存在$[a,b]$使得$x_0\in[a,b]$,由上述讨论知x_0是连续点，由x_0 的任意性，即知积分在$(0,+\infty)$上连续.

七、 $\ln\left(\dfrac{1+a\sin x}{1-a\sin x}\right)\dfrac{1}{\sin x}=2\int_0^a\dfrac{\mathrm{d}y}{1-y^2\sin^2 x}$,

$\dfrac{1}{1-y^2\sin^2 x}$在区域$[0,\frac{\pi}{2}]\times[0,a]$连续，可以交换积分顺序。

$$\int_0^{\frac{\pi}{2}}\ln\left(\frac{1+a\sin x}{1-a\sin x}\right)\frac{\mathrm{d}x}{\sin x}=2\int_0^{\frac{\pi}{2}}\int_0^a\frac{\mathrm{d}y}{1-y^2\sin^2 x}\mathrm{d}x=2\int_0^a\left(\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}x}{1-y^2\sin^2 x}\right)\mathrm{d}y.$$

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}x}{1-y^2\sin^2 x}=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}x}{\sin^2 x(1-y^2)+\cos^2 x}=-\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\cot x}{(1-y^2)+\cot^2 x}\\
&=-\frac{1}{\sqrt{1-y^2}}\arctan\frac{\cot x}{\sqrt{1-y^2}}\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{2\sqrt{1-y^2}}
\end{aligned}$$

$$\int_0^{\frac{\pi}{2}} \ln\left(\frac{1+a\sin x}{1-a\sin x}\right)\frac{\mathrm{d}x}{\sin x} = \int_0^a \frac{\pi}{\sqrt{1-y^2}}\mathrm{d}y = \pi \arcsin a$$

八、 (1) 作变量代换$t=\sin^2 x$,

$\displaystyle\int_0^{\frac{\pi}{2}} \sin^6 x\cos^5 x\mathrm{d}x = \frac{1}{2}\int_0^1 t^{\frac{5}{2}}(1-t)^2\mathrm{d}t = \frac{1}{2}B(\frac{7}{2},3) = \frac{8}{693}.$

(2)记 $\displaystyle\int_0^x f(t)\mathrm{d}t = g(x)$, 三重积分化为

$$\iiint_V f(x)f(y)f(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^{\frac{\pi}{2}} f(x)\left(\int_0^x f(y)\mathrm{d}y\right)\left(\int_0^x f(z)\mathrm{d}z\right)\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{2}} g^2(x)\mathrm{d}g(x) = \frac{1}{3}g^3(x)\Big|_0^{\frac{\pi}{2}} = \frac{1}{3}\left(\frac{8}{693}\right)^3.$$