2023 ～ 2024 学年数分B1期末考试参考答案

一、计算下列各题 **(每小题 6 分, 共 36 分)**

(1) 求不定积分 $\int |x|\ \mathrm{d}x$.

解：原式$=\begin{cases} \frac{1}{2}x^2, & x \geqslant 0 \\ -\frac{1}{2}x^2, & x < 0 \end{cases} + C.$

(2) 求不定积分 $\int \dfrac{x+1}{x^2(x-1)}\ \mathrm{d}x$.

解：由$\frac{x+1}{x^2(x-1)} = \frac{-2}{x} + \frac{-1}{x^2} + \frac{2}{x-1}$，原式$= -2\ln|x| + \frac{1}{x} + 2\ln|x-1| + C.$

(3) 求定积分 $\int_{-1}^{1} \dfrac{x^2+x^3}{1+\sqrt{1-x^2}}\ \mathrm{d}x$.

解：原式$= \displaystyle\int_{-1}^{1} \frac{x^2}{1+\sqrt{1-x^2}}\mathrm{d}x = 2\int_0^1 \frac{x^2\left(1-\sqrt{1-x^2}\right)}{1-(1-x^2)}\mathrm{d}x = 2-2\int_0^{\frac{\pi}{2}} \cos^2 x\mathrm{d}x = 2 - \frac{\pi}{2}.$

(4) 求极限 $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{\pi}{4n}\sec^2\left(\frac{k\cdot\pi}{4n}\right)$.

解：原式$= \displaystyle\int_0^{\frac{\pi}{4}} \sec^2 x\mathrm{d}x = \tan x \Big|_0^{\frac{\pi}{4}} = 1.$

(5) $\displaystyle\int_0^{+\infty} x^3 e^{-x^2}\ \mathrm{d}x$.

解：原式$= \frac{1}{2}\displaystyle\int_0^{+\infty} te^{-t}\mathrm{d}t = \frac{1}{2}.$

(6) 求$(1+x^2)\ln(1+x^2)$的Maclaurin级数，并求其收敛半径.

解：原式$= (1+x^2)\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{n+1}x^{2(n+1)} = x^2 + \sum\limits_{n=2}^{\infty} \frac{(-1)^n}{(n-1)n}x^{2n}.$

收敛半径$R = \lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{1/(n-1)n}} = 1.$

二、(10 分) 求常微分方程$y'' - 3y' + 2y = e^x$的通解.

解：对应齐次方程的特征方程为$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 2$. 故齐次方程的通解为 $y_h = c_1e^x + c_2e^{2x}$.

根据非齐次项特点，可设原方程有特解 $y^* = x\cdot(ce^x)$. 代入得：

$$ce^x\left(x + 2 - 3(x+1) + 2x\right) = e^x \Rightarrow c = -1 \Rightarrow y^* = -xe^x.$$

故所求通解为$y = y_h + y^* = c_1e^x + c_2e^{2x} - xe^x, c_1, c_2$为任意常数.

三、(12 分) 已知曲线$y=\dfrac{e^x+e^{-x}}{2}$ $(0\le x\le 1)$.

(1)求曲线的长度；

(2)求由给定曲线和直线 $x=0,\ x=1, y=0$围成的平面图形绕X轴旋转一周所得立体的体积.

解：(1)$s=\displaystyle\int_0^1\sqrt{1+y'^2(x)}\mathrm{d}x=\int_0^1\sqrt{1+\left(\tfrac{e^x-e^{-x}}{2}\right)^2}\mathrm{d}x=\int_0^1\frac{e^x+e^{-x}}{2}\mathrm{d}x=\frac{e^2-1}{2e}$.

(2)$V=\displaystyle\int_0^1\pi y^2\mathrm{d}x=\pi\int_0^1\left(\frac{2+e^{2x}+e^{-2x}}{4}\right)\mathrm{d}x=\frac{\pi}{8}\left(4+e^2-e^{-2}\right)$.

四、(本题 10 分) 求幂级数 $\sum\limits_{n=1}^{\infty}\dfrac{x^n}{2n-1}$的收敛域与和函数.

解：收敛半径为$\frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{1/2n-1}}=1$, 收敛域为$[-1,1)$. 设$S(x)=\sum\limits_{n=1}^{\infty}\dfrac{x^n}{2n-1}$.

$x\in[0,1)$时，令$x=t^2(t\ge 0)$，则

$$S(t^2)=t\sum_{n=1}^{\infty}\frac{t^{2n-1}}{2n-1}\Rightarrow\left(\frac{S(t^2)}{t}\right)'=\sum_{n=1}^{\infty}t^{2n-2}=\frac{1}{1-t^2}.$$

由$S'(t^2)|_{t=0}=0$可得：$S(t^2)=t\displaystyle\int_0^t\frac{1}{1-u^2}\mathrm{d}u=\frac{t}{2}\ln\frac{1+t}{1-t}\Rightarrow S(x)=\frac{\sqrt{x}}{2}\ln\frac{1+\sqrt{x}}{1-\sqrt{x}}$;

$x\in[-1,0)$时，令$x=-t^2(t>0)$，则

$$S(-t^2)=t\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}t^{2n-1}\Rightarrow\left(\frac{S(-t^2)}{t}\right)'=\sum_{n=1}^{\infty}(-1)^n t^{2n-2}=-\frac{1}{1+t^2}.$$

由$S'(t^2)|_{t=0}=0$可得：$S(-t^2)=-t\arctan t\Rightarrow S(x)=-\sqrt{-x}\arctan\sqrt{-x}$.

综合得，和函数为$S(x)=\begin{cases}\frac{\sqrt{x}}{2}\ln\frac{1+\sqrt{x}}{1-\sqrt{x}}, & x\in[0,1)\\ -\sqrt{-x}\arctan\sqrt{-x}, & x\in[-1,0)\end{cases}$.

五、(8 分) 设$f(x)$在$[0,1]$上二阶导函数连续，$f(0)=f(1)=0$. 证明：

$$\int_0^1\left|f''(x)\right|\,\mathrm{d}x\ge 4\max_{x\in[0,1]}|f(x)|.$$

证明：设$|f(x)|$的最大值点为x_0，不妨设$x_0\ne 0,1$（否则$f(x)\equiv 0$，结论显然成立）.
由Lagrange中值定理，$\exists\xi_1\in(0,x_0)$, $\xi_2\in(x_0,1)$, s.t.

$$\begin{cases}f(x_0)-f(0)=f'(\xi_1)\,x_0\\ f(1)-f(x_0)=f'(\xi_2)\,(1-x_0)\end{cases}\Rightarrow\begin{cases}f'(\xi_1)=\frac{f(x_0)}{x_0}\\ f'(\xi_2)=\frac{-f(x_0)}{1-x_0}\end{cases}$$

于是$\displaystyle\int_0^1|f''(x)|\mathrm{d}x|\geqslant\int_{\xi_1}^{\xi_2}|f''(x)|\mathrm{d}x\geqslant\left|\int_{\xi_1}^{\xi_2}f''(x)\mathrm{d}x\right|=|f'(\xi_2)-f'(\xi_1)|$

$=\left|\frac{f(x_0)}{x_0}+\frac{f(x_0)}{1-x_0}\right|=\frac{|f(x_0)|}{x_0(1-x_0)}\geqslant 4\,|f(x_0)|=4\max\limits_{x\in[0,1]}|f(x)|$.

六、(14分) 设 $u_n=\displaystyle\int_0^1\frac{\mathrm{d}t}{(1+t^2)^n}(n\geqslant 1)$.

(1) 证明数列 $\{u_n\}$ 收敛, 并求极限 $\lim\limits_{n\to\infty}u_n$;

(2) 证明级数 $\sum\limits_{n=1}^{\infty}(-1)^n u_n$ 条件收敛;

(3) 证明当$p\geq 1$时级数$\sum\limits_{n=1}^{\infty}\frac{u_n}{n^p}$ 收敛，并求级数$\sum\limits_{n=1}^{\infty}\frac{u_n}{n}$和.

证明：(1)显然u_n单调递减且有下界0，故$\{u_n\}$收敛. (2分)

$\forall\varepsilon>0,\ \exists N\in\mathbb{N}$, s.t. 当$n>N$时，$\frac{1}{(1+\varepsilon^2)^n}<\varepsilon$. 故当$n>N$时，

$$0<u_n=\int_0^{\varepsilon}\frac{\mathrm{d}t}{(1+t^2)^n}+\int_{\varepsilon}^1\frac{\mathrm{d}t}{(1+t^2)^n}<\varepsilon+\frac{1-\varepsilon}{(1+\varepsilon^2)^n}<\varepsilon+\frac{1}{(1+\varepsilon^2)^n}<2\varepsilon.$$

即$\lim\limits_{n\to\infty}u_n=0$.

注：在成功计算极限的情形下，第一句可以是多余的.

(2)$\{u_n\}$单调递减趋于0，由Leibniz判别法知$\sum\limits_{n=1}^{\infty}(-1)^n u_n$ 收敛;且当$n\geq 2$时,

$$u_n=\int_0^1\frac{\mathrm{d}t}{(1+t^2)^n}\geqslant\int_0^1\frac{\mathrm{d}t}{(1+t)^n}=\frac{1}{n-1}\left(1-2^{1-n}\right)\sim\frac{1}{n}(n\to\infty).$$

$\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$发散，由比较判别法知$\sum\limits_{n=1}^{\infty}u_n$发散. 综合得，$\sum\limits_{n=1}^{\infty}(-1)^n u_n$ 条件收敛.

(3)由比较判别法知，只需证明$\sum\limits_{n=1}^{\infty}\dfrac{u_n}{n}$收敛.

$u_n=\displaystyle\int_0^1\frac{1}{(1+t^2)^n}\mathrm{d}t=\left.\frac{t}{(1+t^2)^n}\right|_0^1-\int_0^1 t\cdot(-n)\cdot\frac{2t}{(1+t^2)^{n+1}}\mathrm{d}t=\frac{1}{2^n}+2n\int_0^1\frac{t^2+1-1}{(1+t^2)^{n+1}}\mathrm{d}t$

$=\dfrac{1}{2^n}+2n\left(u_n-u_{n+1}\right)\Rightarrow u_{n+1}=\left(1-\dfrac{1}{2n}\right)u_n+\dfrac{1}{2n\cdot 2^n}\Rightarrow u_{n+1}-u_n=-\dfrac{u_n}{2n}+\dfrac{1}{2n\cdot 2^n}$.

$\Rightarrow u_{n+1}-u_1=-\dfrac{1}{2}\sum\limits_{k=1}^{n}\dfrac{u_k}{k}+\dfrac{1}{2}\cdot\sum\limits_{k=1}^{n}\dfrac{1}{k\cdot 2^k}$.

令$n\to\infty$，并由$\sum\limits_{k=1}^{\infty}\dfrac{1}{k\cdot 2^k}=\ln 2$得$\sum\limits_{k=1}^{\infty}\dfrac{u_k}{k}=2u_1+\ln 2=\dfrac{\pi}{2}+\ln 2$.

七、(10 分)证明: (1) 函数项级数$\sum\limits_{n=1}^{\infty} \frac{(-x\ln x)^n}{n!}$在$(0,1]$上一致收敛.

$$(2) \int_0^1 x^{-x}\, \mathrm{d}x = \sum_{n=1}^{\infty} \frac{1}{n^n}.$$

证明：(1) $\max\limits_{x\in(0,1]} |x\ln x| = \frac{1}{e}$. 因为$\left|\frac{(-x\ln x)^n}{n!}\right| \leqslant \frac{1}{e^n n!}$且$\sum\limits_{n=1}^{\infty} \frac{1}{e^n n!}$收敛，由Weierstrass判别法知$\sum\limits_{n=1}^{\infty} \frac{(-x\ln x)^n}{n!}$在$(0,1]$上一致收敛.

(2)$\displaystyle\int_0^1 x^{-x}\, \mathrm{d}x = \int_0^1 \mathrm{e}^{-x\ln x}\, \mathrm{d}x = \int_0^1 \left(\sum_{n=0}^{\infty} \frac{1}{n!}(-x\ln x)^n\right) \mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^n \ln^n x\, \mathrm{d}x.$
积分与无穷求和可交换次序是因为函数项级数在$[0,1]$上 的一致收敛性.
令$I_{m,n} = \displaystyle\int_0^1 x^m \ln^n x\, \mathrm{d}x$，则

$$I_{m,n} = \int_0^1 \frac{\ln^n x}{m+1} \mathrm{d}\left(x^{m+1}\right) = \frac{1}{m+1}\left(x^{m+1}\ln^n x\Big|_0^1 - \int_0^1 x^{m+1}\frac{n\ln^{n-1}x}{x}\mathrm{d}x\right) = \frac{-n}{m+1} I_{m,n-1}$$

$$= \frac{-n}{m+1}\cdot\frac{-(n-1)}{m+1} I_{m,n-2} = \cdots = \frac{(-1)^n n!}{(m+1)^n} I_{m,0} = \frac{(-1)^n n!}{(m+1)^{n+1}}.$$

故$\displaystyle\int_0^1 x^n \ln^n x\, \mathrm{d}x = I_{n,n} = \frac{(-1)^n n!}{(n+1)^{n+1}}$. 代入得

$$\int_0^1 x^{-x}\, \mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(-1)^n n!}{(n+1)^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{n^n}.$$