4.2.1 Proof: $X_1, \cdots, X_n$ are measurable w.r.t. $\mathcal{G}_n$. $\forall n \geq 1$

then $\mathcal{F}_n = \sigma(X_1, \cdots, X_n) \subseteq \mathcal{G}_n$.

Obviously, $\{\mathcal{F}_n\}_{n\in N}$ is a filtration, and $X_n \in \mathcal{F}_n$.

$E(X_{n+1}|\mathcal{F}_n) = E(E(X_{n+1}|\mathcal{G}_n)|\mathcal{F}_n) = E(X_n|\mathcal{F}_n) = X_n$. $\forall n \geq 1$

$\Rightarrow X_n$ is a martingale w.r.t. $\mathcal{F}_n$

4.2.2 Sol: Let $\Omega = (0,1)$. $\mathcal{F}_n = \{$Lebesgue measurable sets$\}$. P = Lebesgue measure

$\{\mathcal{F}_n\}$ is a filtration. Take $X_n \equiv -\frac{1}{n+1}$. $\forall n \geq 0$

$E(X_{n+1}|\mathcal{F}_n) = -\frac{1}{n+2} > -\frac{1}{n+1} = X_n$. $E(X_{n+1}^2|\mathcal{F}_n) = \frac{1}{(n+2)^2} < \frac{1}{(n+1)^2} = X_n^2$

$\Rightarrow \{X_n\}_{n\geq 0}$ is a submartingale and $\{X_n^2\}_{n\geq 0}$ is a supermartingale.

4.2.3 Proof: $E(X_{n+1} \vee Y_{n+1}|\mathcal{F}_n) \geq E(X_{n+1}|\mathcal{F}_n) \geq X_n$,

$E(X_{n+1} \vee Y_{n+1}|\mathcal{F}_n) \geq E(Y_{n+1}|\mathcal{F}_n) \geq Y_n$

$\Rightarrow E(X_{n+1} \vee Y_{n+1}|\mathcal{F}_n) \geq X_n \vee Y_n$

then $X_n \vee Y_n$ is a submartingale w.r.t. $\mathcal{F}_n$

4.2.4 Proof: 1° $\forall K > 0$. Let $N_K = \inf\{n : X_n \geq K\}$, then N_K is a stopping time.

$H_n = 1_{(N_K \geq n)}$, then H_n is predictable. $H_n \geq 0$

We may set $X_0 \equiv 0$, and since X_n is a submartingale, so is $(H \cdot X)_n = X_{n\wedge N_K}$

2° $X_{n\wedge N_K}^+ = X_n^+ 1_{(n<N_K)} + X_{N_K}^+ 1_{(n \geq N_K)}$

Note that $X_n^+ 1_{(n<N_K)} \leq K$,

$X_{N_K}^+ 1_{(n\geq N_K)} = X_{N_K} 1_{(n \geq N_K)} = (X_{N_K-1} + \xi_{N_K}) 1_{(n\geq N_K)} \leq K + \sup_{n\geq 1} \xi_n^+$

$\Rightarrow X_{n\wedge N_K}^+ \leq 2K + \sup_{n\geq 1} \xi_n^+$

$\Rightarrow EX_{n\wedge N_K}^+ \leq 2K + E(\sup_{n\geq 1} \xi_n^+) < \infty$

By Thm 4.2.11. the submartingale $X_{n\wedge N_K}$ converges a.s.

3° Denote $C = \{\sup_{n\geq 1} X_n < \infty\}$. then $P(C) = 1$

$C = \bigcup_{K=1}^{\infty} \{\sup_{n\geq 1} X_n < K\}$. X_n converges a.s. on $\{\sup_{n\geq 1} X_n < K\}$

$\Rightarrow X_n$ converges a.s.

此题的思路同Thm 4.3.1的证明，都是在集合$D\subset\Omega$上证明某个性质成立．先将D剖分成$\bigcup_{K=1}^{\infty} D_K$，而在$D_K$上可加上一个停时$N_K$，来获得某种“一致有界性”．

4.2.5 Sol: Suppose ξ_i are independent. and $P(\xi_i = -1) = \frac{1}{2i}$, $P(\xi_i = i) = \frac{1}{2i^2}$. $P(\xi_i = 0) = 1 - \frac{1}{2i} - \frac{1}{2i^2}$

$E\xi_i = 0 \Rightarrow X_n = \xi_1 + \cdots + \xi_n$ is a martingale w.r.t. $\mathcal{F}_n = \sigma(\xi_1, \cdots, \xi_n)$

$\sum_{i=1}^{\infty} P(\xi_i > 0) = \sum_{i=1}^{\infty} P(\xi_i = i) = \sum_{i=1}^{\infty} \frac{1}{2i^2} < \infty \Rightarrow P(\xi_i > 0 \text{ i.o.}) = 0$

$\sum_{i=1}^{\infty} P(\xi_i < -\frac{1}{2}) = \sum_{i=1}^{\infty} P(\xi_i = -1) = \sum_{i=1}^{\infty} \frac{1}{2i} = \infty \Rightarrow P(\xi_i < -\frac{1}{2}, \text{ i.o.}) = 1$

Hence. $X_n \to -\infty$ a.s.

4.2.6. (i) By Thm 4.2.12. we find that X_n converges a.s.

$E(Y_m)=1$. $P(Y_m=1)<1$. $\Rightarrow P(Y_m>1)>0 \Rightarrow \exists\, \varepsilon>0$. s.t. $P(Y_m\geq e^{\varepsilon})>0$

$\Rightarrow \sum_{m=1}^{\infty} P(Y_m\geq e^{\varepsilon})=\infty$. since Y_i's are identically distributed.

By Borel-Catelli, $P(Y_m\geq e^{\varepsilon}\ i.o.)=1$.

$\{X_n\to a>0\}\subseteq\{\log Y_m\to 0\}\subseteq\bigcup_{N=1}^{\infty}\{e^{-\varepsilon}<Y_m<e^{\varepsilon}, \forall m>N\}$

$\Rightarrow P(X_n\to a>0)=0$.

$X_n\to 0$. a.s.

(ii) This conclusion could be false due to the case $E\log Y_m=-\infty$.

Here is a counterexample. Take $0<c<<1$.

$P(Y=e^{-n})=\frac{c}{n^2}$. then $\sum_{n=1}^{\infty} e^{-n}\frac{c}{n^2}<1$, and $\sum_{n=1}^{\infty}\frac{c}{n^2}<1$

$P\left(Y=\frac{1-\sum_{n=1}^{\infty} e^{-n}\frac{c}{n^2}}{1-\sum_{n=1}^{\infty}\frac{c}{n^2}}\right)=1-\sum_{n=1}^{\infty}\frac{c}{n^2}$. then $EY=1$

$E|\log Y|\geq\sum_{n=1}^{\infty}\frac{c}{n^2}|\log e^{-n}|=\sum_{n=1}^{\infty}\frac{c}{n}=\infty$. $Y_m=Y$ i.i.d. $(m\geq 1)$

By Strong Law of large numbers, the conclusion is false.

4.2.8 Proof: Let $H_n=X_n/\prod_{i=1}^{n-1}(1+Y_i)$ (noting that Y_n is positive)

then H_n is adapted to $\mathcal{F}_n$

$$E(H_{n+1}|\mathcal{F}_n)=\frac{E(X_{n+1}|\mathcal{F}_n)}{\prod_{i=1}^{n}(1+Y_i)}\leq\frac{X_n}{\prod_{i=1}^{n-1}(1+Y_i)}=H_n$$

$\Rightarrow H_n$ is a supermartingale. $\xRightarrow[4.2.12]{\text{Thm}}$ H_n converges a.s.

$\sum Y_n<\infty$ a.s. $\Rightarrow \prod_{n=1}^{\infty}(Y_n+1)$ converges a.s.

$X_n=H_n\cdot\prod_{i=1}^{n-1}(1+Y_i)$ converges a.s. to a finite limit.

4.2.9. Proof: N is a stopping time $\Rightarrow \{N>n+1\}=\{N\leq n\}^c\in\mathcal{F}_n$

$$E(Y_{n+1}|\mathcal{F}_n)=E(X^1_{n+1}1_{(N>n+1)}|\mathcal{F}_n)+E(X^2_{n+1}1_{(N\leq n+1)}|\mathcal{F}_n)$$

$$=1_{(N>n+1)}E(X^1_{n+1}|\mathcal{F}_n)+1_{(N\leq n+1)}E(X^2_{n+1}|\mathcal{F}_n)$$

$$\leq 1_{(N>n+1)}X^1_n+1_{(N\leq n)}X^2_n+1_{(N=n+1)}E(X^2_{n+1}|\mathcal{F}_n)$$

$$=1_{(N>n+1)}X^1_n+1_{(N\leq n)}X^2_n+1_{(N=n+1)}X^1_n=Y_n$$

It's similar to prove $E(Z_{n+1}|\mathcal{F}_n)\leq Z_n$

4.3.3 Proof: Let $H_n = X_n - \sum_{m=0}^{n-1} Y_m$, then $H_n \in \mathcal{F}_n$. ($H_0 \triangleq X_0$)

$E(H_{n+1}|\mathcal{F}_n) = E(X_{n+1}|\mathcal{F}_n) - \sum_{m=0}^{n} Y_m \leq X_n + Y_n - \sum_{m=0}^{n} Y_m = H_n$

$\Rightarrow (H_n)_{n\geq 0}$ is a supermartingale.

$\forall M>0$. denote $N = \inf\{k\in\mathbb{N} \mid \sum_{m=0}^{k} Y_m > M\}$

$\{N=n\} = \{\sum_{m=0}^{n} Y_m > M, \sum_{m=0}^{k} Y_m \leq M \text{ for } k<n\} \in \mathcal{F}_n$

$\Rightarrow N$ is a stopping time.

$\Rightarrow (H_{N\wedge n})_{n\geq 0}$ is a supermartingale

$H_{N\wedge n} = X_{N\wedge n} - \sum_{m=0}^{N\wedge n-1} Y_m \geq -\sum_{m=0}^{N\wedge n-1} Y_m \geq -M$ (note that $N\wedge n-1 < N$)

By Thm 4.2.12. $\{H_{N\wedge n}\}$ converges, a.s.

$\Rightarrow \{H_n\}$ converges a.s. on the set $\{N=\infty\}$

$\sum_{n=0}^{\infty} Y_n < \infty$ a.s. $\Rightarrow P(\sum_{n=0}^{\infty} Y_n < \infty) = P(\bigcup_{k=1}^{\infty}\{\sum_{n=0}^{\infty} Y_n < k\}) = 1$

$\{H_n\}$ converges a.s. to a finite limit.

$\Rightarrow \{X_n\}$ converges a.s. to a finite limit.

4.3.5 Proof: Denote $a_n = \dfrac{P(\bigcap_{m=1}^{n} A_m^c)}{P(\bigcap_{m=1}^{n-1} A_m^c)}$, then $P(A_n | \bigcap_{m=1}^{n-1} A_m^c) = \dfrac{P(A_n \cap (\bigcap_{m=1}^{n-1} A_m^c))}{P(\bigcap_{m=1}^{n-1} A_m^c)} = 1 - a_n$

$\Rightarrow \sum_{n=2}^{+\infty} (1-a_n) = \infty$, And $a_n = \dfrac{P(\bigcap_{m=1}^{n} A_m^c)}{P(\bigcap_{m=1}^{n-1} A_m^c)} \longrightarrow 1$ provided that $P(\bigcap_{m=1}^{\infty} A_m^c) > 0$.

$\dfrac{-\ln(a_n)}{1-a_n} = \dfrac{\ln(1+a_n-1)}{a_n-1} \to 1$ as $n\to\infty$

$\Rightarrow \sum_{n=2}^{\infty}(-\ln a_n) = \infty \Rightarrow \ln(\prod_{n=2}^{\infty} a_n) = -\infty \Rightarrow \prod_{n=2}^{\infty} a_n = 0$

i.e. $P(\bigcap_{m=1}^{\infty} A_m^c) = C \prod_{n=2}^{\infty} a_n = 0$, contradiction!

Another proof of Thm 4.3.5

Write $\mathcal{F}_n = \sigma(A_1, \cdots, A_n)$

$P(A_n|\mathcal{F}_{n-1})\big|_{\bigcap_{m=1}^{n-1} A_m^c} \equiv P(A_n | \bigcap_{m=1}^{n-1} A_m^c)$

$\Rightarrow \bigcap_{m=1}^{\infty} A_m^c \subseteq \{\sum P(A_n|\mathcal{F}_{n-1}) = \infty\}$

$\overset{a.s.}{\underset{\text{Thm 4.3.4}}{=}} \{A_n \text{ i.o.}\}$

And we conclude that $P(\bigcap_{m=1}^{\infty} A_m^c) = 0$.

4.3.11 Proof: Denote $\theta = P(\lim Z_n/\mu^n = 0) = \sum_{k=0}^{\infty} P(\lim Z_n/\mu^n = 0 | Z_1 = k) P(Z_1 = k)$

Apparently, $P(\lim Z_n/\mu^n > 0 | Z_1 = 0) = 1$

Suppose $k>0$. $Z_2 = \xi_1^1 + \xi_2^1 + \cdots + \xi_k^1$.

Then each ξ_j^1 will deduce another branching process which is independent.

$\Rightarrow P(\lim Z_n/\mu^n = 0 | Z_1 = k) = \theta^k$

i.e. $\theta = \sum_{k=0}^{\infty} \theta^k p_k = \varphi(\theta)$. and $\theta < 1 \Rightarrow \theta = \rho$

Z_n/μ^n converges a.s.

$\Rightarrow \{\lim Z_n/\mu^n > 0\} = \{\lim Z_n/\mu^n = 0\}^c \overset{a.s.}{=} \{Z_n = 0 \text{ for some } n\}^c = \{Z_n > 0 \text{ all the time}\}$

4.3.12 Proof: On $\{Z_n = k\}$. $E(\rho^{Z_{n+1}} | \mathcal{F}_n) = E(\prod_{i=1}^{k} \rho^{\xi_i^{n+1}} | \mathcal{F}_n) = E(\prod_{i=1}^{k} \rho^{\xi_i^{n+1}}) = \prod_{i=1}^{k} E(\rho^{\xi_i^{n+1}})$

$= (\sum_{j\geq 0} \rho^j p_j)^k = \varphi(\rho)^k = \rho^{Z_n}$

$\Rightarrow E(\rho^{Z_{n+1}} | \mathcal{F}_n) = \rho^{Z_n}$ a.s. i.e. $(\rho^{Z_n})_{n\geq 0}$ is a martingale

Suppose $Z_0 = x$; there are x independent branching processes. $\Rightarrow P(Z_n = 0 \text{ for some } n | Z_0 = x) = \rho^x$

4.4.1 Proof: $\{N=j\}\in\mathcal{F}_j$

$$E(X_j;N=j)\le E(E(X_k|\mathcal{F}_j);N=j)=E(X_k;N=j)$$

$$EX_N=\sum_{j=0}^{k}E(X_j;N=j)\le\sum_{j=0}^{k}E(X_k;N=j)=EX_k$$

4.4.2 Proof: $H_n=1_{(M<n\le N)}\in\mathcal{F}_n$. $(X_n)_{n\ge0}$ is a submartingale.

$\Rightarrow ((H\cdot X)_n)_{n\ge0}$ is also a submartingale.

Because $P(N\le k)=1$, then $E(H\cdot X)_k=E(\sum_{j=1}^{k}1_{(M<j\le N)}(X_j-X_{j-1}))$

$$=E(X_N-X_M)\ge E(H\cdot X)_0=0$$

$\Rightarrow EX_M\le EX_N$

4.4.3 Proof: $\mathcal{F}_M=\{A\in\mathcal{A}\mid A\cap\{M\le n\}\in\mathcal{F}_n,\ \forall n\in\mathbb{N}\}$

$$\forall A\in\mathcal{F}_M.\ A\cap\{N\le n\}=\bigcup_{k=0}^{n}(A\cap\{N=k\})\xlongequal{M\le N}\bigcup_{k=0}^{n}\Big(A\cap\big(\bigcup_{m=0}^{k}\{N=k\}\cap\{M=m\}\big)\Big)$$

$$=\bigcup_{k=0}^{n}\bigcup_{m=0}^{k}(A\cap\{M=m\}\cap\{N=k\})\in\mathcal{F}_n,\ \forall n\in\mathbb{N}$$

$\Rightarrow A\in\mathcal{F}_N$, i.e. $\mathcal{F}_M\subset\mathcal{F}_N$

$\forall n\in\mathbb{N}$. $\{L=n\}=(\{M=n\}\cap A)\cup(\{N=n\}\cap A^c)$

By definition, $\{M=n\}\cap A\in\mathcal{F}_n$.

$A\in\mathcal{F}_N\Rightarrow\{N=n\}\cap A^c=\{N=n\}-\{N=n\}\cap A\in\mathcal{F}_n$

Then $\{L=n\}\in\mathcal{F}_n$. $\Rightarrow L$ is a stopping time.

4.4.4 Proof: $\forall A\in\mathcal{F}_M$. let $L=M1_A+N1_{A^c}$

In 4.4.3, we have checked that L is a stopping time. And $L\le N$, $P(N\le k)=1$

$\xRightarrow{4.4.2} EX_L\le EX_N$

i.e. $E(X_M;A)+E(X_N;A^c)\le EX_N\Rightarrow E(X_M;A)\le E(X_N;A)$

Then $X_M\le E(X_N|\mathcal{F}_M)$

4.4.5 Proof: $E[E(Y|\mathcal{G})E(Y|\mathcal{F})]=E[E(YE(Y|\mathcal{F})|\mathcal{G})]=E[YE(Y|\mathcal{F})]$

$$=E[E(YE(Y|\mathcal{F})|\mathcal{F})]=E[E(Y|\mathcal{F})E(Y|\mathcal{F})]$$

$$\Rightarrow E(E[Y|\mathcal{G}]-E[Y|\mathcal{F}])^2=E(E[Y|\mathcal{G}])^2-2E[E(Y|\mathcal{G})E(Y|\mathcal{F})]+E(E[Y|\mathcal{F}])^2$$

$$=E(E[Y|\mathcal{G}])^2-E(E[Y|\mathcal{F}])^2$$

4.4.6 Proof: Denote $A=\{\max_{1\le m\le n}|S_m|>x\}$. $N=\inf\{m:|S_m|>x\text{ or }m=n\}$

N is a stopping time, $P(N\le n)=1$, $S_n^2-s_n^2$ is a martingale

$\xRightarrow{\text{Thm 4.4.1}} 0=E(S_N^2-s_N^2)\le(x+K)^2P(A)+(x^2-var(S_n))P(A^c)$

Write $P(A)=1-P(A^c)$

$\Rightarrow (x+K)^2\ge(var(S_n)-x^2+(x+K)^2)P(A^c)\ge var(S_n)P(A^c)$, i.e. $P(\max_{1\le m\le n}|S_m|\le x)\le\dfrac{(x+K)^2}{var(S_n)}$

4.4.7 Proof: X_n is a martingale $\Rightarrow (X_n+c)^2$ is a submartingale for any $c\in\mathbb{R}$

$c+\lambda>0$.

$$P(\max_{1\le m\le n} X_m\ge\lambda)\le P(\max_{1\le m\le n}(X_m+c)^2\ge(c+\lambda)^2)\le\frac{E(X_n+c)^2}{(c+\lambda)^2}=\frac{EX_n^2+c^2}{(c+\lambda)^2}$$

Take $c=\frac{EX_n^2}{\lambda}$.

$$P(\max_{1\le m\le n} X_m\ge\lambda)\le\frac{EX_n^2}{EX_n^2+\lambda^2}$$

4.4.9 Proof: $m\ge1$. $E(X_m-X_{m-1})(Y_m-Y_{m-1})=EX_mY_m-E(X_mY_{m-1}+X_{m-1}Y_{m-1}-X_{m-1}Y_{m-1})$

$$=EX_mY_m+EX_{m-1}Y_{m-1}-E(Y_{m-1}E(X_m|\mathcal{F}_{m-1})+X_{m-1}E(Y_m|\mathcal{F}_{m-1}))$$

$$=EX_mY_m-EX_{m-1}Y_{m-1}$$

$$\Rightarrow EX_nY_n-EX_0Y_0=\sum_{m=1}^n E(X_m-X_{m-1})(Y_m-Y_{m-1})$$

4.4.10 Proof: $M=EX_0^2+\sum_{m=1}^\infty E\xi_m^2<\infty$

By 4.4.9, $E\xi_m^2=E(X_m-X_{m-1})^2=EX_m^2-EX_{m-1}^2$

$$\Rightarrow EX_n^2=EX_0^2+\sum_{m=1}^n E\xi_m^2\le M,\ \forall n\ge1$$

$\xRightarrow[4.4.6]{\text{Thm}}$ $X_n\to X_\infty$ a.s. and in L^2.

4.6.4. Proof: $\{\lim_{n\to\infty}X_n=\infty\}^c=\bigcup_{N=1}^\infty\{X_n\le N.\ \text{i.o.}\}$

$\forall\omega\in\{X_n\le N.\ \text{i.o.}\}$. there exists a subsequence $X_{n(k)}$ s.t. $X_{n(k)}(\omega)\le N$

$P(D|X_1,\cdots X_{n(k)})\ge\delta(N)>0$. $\forall k\in\mathbb{N}$

By thm 4.6.9. $P(D|X_1,\cdots X_{n(k)})\to 1_D$ as $k\to\infty$ a.s.

$\Rightarrow 1_D\ge\delta(N)>0$ a.s. on $\{X_n\le N.\ \text{i.o.}\}$

$\Rightarrow 1_D=1$ a.s. on $\{\lim_{n\to\infty}X_n=\infty\}^c$, i.e. $P(D\cup\{\lim_{n\to\infty}X_n=\infty\})=1$

4.6.5 Proof: Let $D=\{Z_n=0$ for some $n\}$, $P(D)\ge P(Z_1=0)=p_0>0$ (Assume $Z_0=1$)

$0<P(D)\le1$

Suppose $Z_n=k$, then $\xi_1^n,\xi_2^n,\cdots,\xi_k^n$ can be seen as k independent branching processes. It follows that $P(D|Z_1,\cdots,Z_n)=P(D)^k$ on $\{Z_n=k\}$

$$\Rightarrow P(D|Z_1,\cdots,Z_n)\ge P(D)^x>0 \text{ a.s. on } \{Z_n\le x\}$$

By Exercise 4.6.4, $P(D\cup\{\lim_{n\to\infty}Z_n=\infty\})=1$. i.e. $P(\lim_{n\to\infty}Z_n=0$ or $\infty)=1$

4.6.7 Proof: $\{E(Y|\mathcal{F}_n)\}_{n\ge0}$ is a uniformly integrable martingale. And $\mathcal{F}_n\uparrow\mathcal{F}_\infty$

$$\Rightarrow E(Y|\mathcal{F}_n)\to E(Y|\mathcal{F}_\infty) \text{ in } L^1$$

$\forall\varepsilon>0$. $\exists N\in\mathbb{N}$. s.t. $E|E(Y|\mathcal{F}_n)-E(Y|\mathcal{F}_\infty)|<\frac{\varepsilon}{2}$, $E|Y-Y_n|<\frac{\varepsilon}{2}$, $\forall n\ge N$

$$E|E(Y_n|\mathcal{F}_n)-E(Y|\mathcal{F}_\infty)|\le E|E(Y|\mathcal{F}_n)-E(Y|\mathcal{F}_\infty)|+E|E(Y_n-Y|\mathcal{F}_n)|$$

$$\le\frac{\varepsilon}{2}+E(E(|Y_n-Y||\mathcal{F}_n))=\frac{\varepsilon}{2}+E|Y-Y_n|\le\varepsilon\Rightarrow E(Y_n|\mathcal{F}_n)\xrightarrow{L^1}E(Y|\mathcal{F}_\infty)$$

4.7.1 Proof: $\{X_n\}_{n\le 0}$ is a backwards martingale. $p>1$. Assume $X_n \to X_{-\infty}$ a.s.

$\forall n\ge 0$, $\{X_{-n}, X_{-(n-1)}, \cdots, X_0\}$ is a martingale.

Applying Thm 4.4.4 we have

$$E|X_{-n}^*|^p \le \left(\frac{p}{p-1}\right)^p E|X_0|^p, \text{ where } X_{-n}^* = \max_{-n\le m\le 0} |X_m|$$

And X_{-n}^* increases as $n\to\infty$.

$$\Rightarrow E\left(\sup_{n\le 0}|X_n|\right)^p \le \left(\frac{p}{p-1}\right)^p E|X_0|^p < \infty.$$

$X_{-n}\to X_{-\infty}$ a.s., and $|X_{-n}|^p \le \sup|X_n|^p$

By dominated convergence, the backwards martingale converges in L^p.

4.7.2 Proof: Let $W_N = \sup\{|Y_n - Y_m| : n, m \le -N\}$. Since $Y_n \to Y_{-\infty}$ a.s. as $n\to-\infty$,

W_N decreases a.s. to 0 as $N\to+\infty$. $W_N \le 2Z$. then $EW_N<\infty$

$\Rightarrow E(W_N|\mathcal{F}_{-\infty}) \to 0$ a.s. as $N\to+\infty$

$$\limsup_{n\to-\infty} E(|Y_n - Y_{-\infty}|\,|\mathcal{F}_n) \le \lim_{n\to-\infty} E(W_N|\mathcal{F}_n) = E(W_N|\mathcal{F}_{-\infty}) \downarrow 0 \text{ as } N\to+\infty \text{ a.s.}$$

$$\Rightarrow \limsup_{n\to-\infty} E(|Y_n - Y_{-\infty}|\,|\mathcal{F}_n) = 0. \text{ a.s. as } n\to-\infty$$

By Thm 4.7.1. $E(Y_{-\infty}|\mathcal{F}_n) \to E(Y_{-\infty}|\mathcal{F}_{-\infty})$ a.s. as $n\to-\infty$

$\Rightarrow E(Y_n|\mathcal{F}_n) \to E(Y_{-\infty}|\mathcal{F}_{-\infty})$ a.s. as $n\to-\infty$

4.8.1 Proof: We can apply Thm 4.8.1 to the uniformly integrable submartingale $Y_{M\wedge n}$ and the stopping time L, then we get

$\{Y_{M\wedge(L\wedge n)}\}_{n\ge 0}$ is uniformly integrable.

$L\wedge n \le L \le M \Rightarrow Y_{M\wedge(L\wedge n)} = Y_{L\wedge n}$

Then $Y_{L\wedge n} \to Y_L$ in L^1 and $Y_{M\wedge n}\to Y_M$ in L^1

$EY_{L\wedge n} \le EY_{M\wedge n}$, $\forall n\in\mathbb{N}$. Let $n\to\infty$, then $EY_L \le EY_M$

$\forall A\in\mathcal{F}_L (\subset \mathcal{F}_M)$, set $T = L1_A + M1_{A^c} \le M$. T is a stopping time,

and it follows that $EY_T \le EY_M \Rightarrow E(Y_L; A) \le E(Y_M; A)$. $\forall A\in\mathcal{F}_L$

It follows $Y_L \le E(Y_M|\mathcal{F}_L)$

4.8.2 Proof: X_n is a nonnegative supermartingale, so it converges a.s. to X_∞. $X_\infty\ge 0$ a.s.

Set $N = \inf\{n : X_n > \lambda\} \le \infty$, N is a stopping time. $X_N > \lambda$ on $\{\sup X_n > \lambda\}$

Applying Thm 4.8.4, we have

$$EX_0 \ge EX_N \ge \lambda P(\sup X_n > \lambda) \Rightarrow P(\sup X_n > \lambda) \le EX_0/\lambda$$

4.8.3 Proof: Suppose $ET=\infty$, then $ET\geq a^2/\sigma^2$. Otherwise, we assume $ET<\infty$.

$\Rightarrow T<\infty$ a.s. $\Rightarrow \sup|S_n|>a$ a.s.

$S_n^2-n\sigma^2$ is a martingale, so is $S_{T\wedge n}^2-(T\wedge n)\sigma^2$.

$\Rightarrow \sigma^2 E(T\wedge n)=ES_{T\wedge n}^2$, $\forall n\in\mathbb{N}$

$T\wedge n\uparrow T$ as $n\to\infty$, by monotone convergence,

$\sigma^2 ET=\lim_{n\to\infty}\sigma^2E(T\wedge n)=\lim_{n\to\infty}ES_{T\wedge n}^2\geq E\lim_{n\to\infty}S_{T\wedge n}^2=ES_T^2\geq a^2$ ($T<\infty$ a.s.)

$\Rightarrow ET\geq a^2/\sigma^2$. (Fatou's Lemma)

4.8.4 Proof: $\{S_{T\wedge n}^2-\sigma^2(T\wedge n)\}_{n\geq0}$ is a martingale

$\Rightarrow E(S_{T\wedge n}^2-S_{T\wedge(n-1)}^2)=\sigma^2E[(T\wedge n)-(T\wedge(n-1))]=\sigma^2P(T\geq n)$

$\Rightarrow ES_{T\wedge n}^2=\sum_{m=1}^n\sigma^2P(T\geq m)\Rightarrow \sup_n ES_{T\wedge n}^2=\sigma^2ET<\infty$

Applying Thm 4.4.6 to the martingale $\{S_{T\wedge n}\}_{n\geq0}$, we have $S_{T\wedge n}\to S_T$ a.s. and in L^2.

$\sigma^2E(T\wedge n)=ES_{T\wedge n}^2$. $T\wedge n\uparrow T$ as $n\to\infty$

$\Rightarrow \sigma^2ET=ES_T^2$

4.8.5 Proof: (a) $E_xV_0=\frac{x}{1-2p}<\infty\Rightarrow V_0<\infty$ a.s. $\Rightarrow S_{V_0}=0$ a.s. $S_0=x>0$

$\eta_i=\xi_i-(p-q)$, then $\eta_1,\cdots$ independent. $T_n=\sum_{i=1}^n\eta_i$. $E\eta_i=0$. $var(\eta_i)=1-(p-q)^2$

Applying Exercise 4.8.4, we have $(1-(p-q)^2)EV_0=ET_{V_0}^2$.

$T_n=S_n-x-n(p-q)\Rightarrow T_{V_0}=S_{V_0}-x-V_0(p-q)=-x-V_0(p-q)$

i.e. $(1-(p-q)^2)EV_0=E(x+(p-q)V_0)^2=x^2+2(p-q)xEV_0+(p-q)^2EV_0^2$

$\Rightarrow var(V_0)=EV_0^2-(EV_0)^2=x\frac{1-(p-q)^2}{(q-p)^3}-\frac{x^2}{(q-p)^2}$

(b) Actually $var_x(V_0)$ isn't linear w.r.t. x, but $E_xV_0^2$ is.

4.8.7. Proof: 1° $E(S_{n+1}^4-S_n^4|\mathcal{F}_n)=E(4S_n^3\xi_{n+1}+6S_n^2\xi_{n+1}^2+4S_n\xi_{n+1}^3+\xi_{n+1}^4|\mathcal{F}_n)=6S_n^2+1$

$\Rightarrow E([S_{n+1}^4-6(n+1)S_{n+1}^2]-[S_n^4-6nS_n^2]|\mathcal{F}_n)=1-6(n+1)E(S_{n+1}^2-S_n^2|\mathcal{F}_n)=-6n-5$

If we set $b=3$, $c=2$, then

$Y_n=S_n^4-6nS_n^2+3n^2+2n$ is a martingale.

2° $Y_{T\wedge n}$ is also a martingale. $T=\min\{n\geq0: S_n\notin(-a,a)\}$

$ET=a^2$ by Thm 4.8.7. $\{S_{T\wedge n}\}$ bounded $\Rightarrow S_{T\wedge n}\to S_T$ in L^4 and a.s.

$ET<\infty\Rightarrow T<\infty$ a.s. $\Rightarrow S_T^2=a^2$ a.s.

$EY_T=0\Rightarrow 3ET^2=6ETS_T^2-ES_T^4-2ET=5a^4-2a^2\Rightarrow ET^2=\frac{5}{3}a^4-\frac{2}{3}a^2$

4.8.8 Proof: $X_{n\wedge\tau} \to X_\tau$ a.s. $\xRightarrow{\text{Fatou's Lemma}}$ $EX_\tau \le \liminf EX_{n\wedge\tau} = 1 < \infty$

$0 \le X_n 1_{(\tau>n)} \le \exp(\theta_0 a)$ bounded $\Rightarrow \{X_n 1_{(\tau>n)}\}_{n\ge0}$ is uniformly integrable.

$\xRightarrow{\text{Thm 4.8.2}}$ $\{X_{n\wedge\tau}\}_{n\ge0}$ is uniformly integrable.

$EX_\tau = \lim_{n\to\infty} EX_{n\wedge\tau} = 1$

$X_\tau \ge \exp(\theta_0 a)$ on $\{S_\tau \le a\}$, then $1 = EX_\tau \ge \exp(\theta_0 a) P(S_\tau \le a)$

$\Rightarrow P(S_\tau \le a) \le \exp(-\theta_0 a)$

4.8.9 Proof: Assume $\theta_0 < 0$, and $E\exp(\theta_0 \xi_i) = 1$. Then $X_n = \exp(\theta_0 S_n)$ is a non-negative martingale.

Note that $P(\xi_i < -1) = 0$ and S_n is integer-valued,

then $n < T$ implies $S_n > a$ i.e. $X_n 1_{\{n<T\}} < \exp(\theta_0 a)$ $\xRightarrow{\text{Thm 4.8.2}}$ $\{X_{n\wedge T}\}$ is uniformly integrable.

~~Let $b > 0 > a$, $L_b = \inf\{n\ge0 : S_n \ge b\}$. $N = L_b \wedge T = \inf\{n\ge0 : S_n \notin (a,b)\}$.~~

~~$\{X_{n\wedge N}\}_{n\ge0}$ is also a martingale. $X_{n\wedge N} \to X_N$ a.s.~~

~~$S_{n\wedge N} \ge a \Rightarrow X_{n\wedge N} \le \exp(\theta_0 a)$ bounded $\Rightarrow EX_{n\wedge N} \to EX_N$ as $n\to\infty$~~

~~$\Rightarrow EX_N = \lim_{n\to\infty} EX_{n\wedge N} = EX_0 = 1$~~

~~$S_N = a$ on $\{T < L_b\}$; $S_N \ge b$ on $\{T > L_b\}$~~

~~$\Rightarrow 1 = EX_N = \exp(\theta_0 a) P(T < L_b) + EX_N 1_{(T>L_b)}$~~ (*)

~~$0 \le EX_N 1_{(T>L_b)} \le \exp(\theta_0 b) \to 0$ as $b \to +\infty$.~~

~~$T < \infty \Leftrightarrow T < L_b$ for some b~~

~~Let $b\to\infty$ in (*), we have $P(T<\infty) = \exp(-\theta_0 a)$~~

$X_{n\wedge T} \to X_T$ in L^1

$EX_T = EX_0 = 1$

$\frac{S_n}{n} \to E\xi_i > 0$ a.s. by the strong law of large numbers

$\Rightarrow X_\infty = 0$ a.s.

$\Rightarrow X_T = \exp(\theta_0 a)$ on $\{T < \infty\}$

$X_T = 0$ on $\{T = \infty\}$

$1 = EX_T$, then $P(T<\infty) = \exp(-\theta_0 a)$

4.8.10 Proof: Suppose $\theta_0 = \ln(\sqrt{2}-1) < 0$, then

$$E\exp(\theta_0 \xi_j) = \frac{e^{-\theta_0} + e^{\theta_0} + e^{2\theta_0}}{3} = \frac{1}{3}(\sqrt{2}+1 + \sqrt{2}-1 + 3 - 2\sqrt{2}) = 1$$

$S_n = \xi_1 + \cdots + \xi_n$, $X_n = \exp(\theta_0 S_n)$ is a martingale.

$T = \inf\{n \ge 0 : S_n = -i\}$

Using the argument in Exercise 4.8.9, $P(T<\infty) = \exp(-\theta_0 \cdot (-i)) = (\sqrt{2}-1)^i$

And that's the probability we ever go broke.

4.8.11 Proof: $\xi_i = c - \zeta_i \sim \text{Normal}(c-\mu, \sigma^2)$. And assume that $\xi_1, \xi_2, \cdots$ are independent.

$R_n = \xi_1 + \xi_2 + \cdots + \xi_n$

$$E\exp(\theta\xi_n) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{\theta x} e^{-\frac{1}{2\sigma^2}(x-c+\mu)^2} dx = e^{\frac{\sigma^2\theta^2}{2} + \theta(c-\mu)} = 1,\ \theta < 0 \Rightarrow \theta = -\frac{2(c-\mu)}{\sigma^2}$$

$X_n = \exp(\theta R_n)$ is a martingale, $X_0 = 1$. $N = \inf\{n \ge 0 : R_n \le -S_0\}$. $X_{n\wedge N} \to X_N$ in L^1.

$1 = EX_N \ge E(X_N; N<\infty) \ge \exp(-\theta S_0) P(N<\infty) \Rightarrow P(\text{ruin}) = P(N<\infty) \le \exp(-2(c-\mu)S_0/\sigma^2)$.

5

5.1.1 Proof: It's easy to check that $P(X_{n+1}=j\,|\,X_n=i)=\begin{cases} i/N, & j=i \\ 1-i/N, & j=i+1 \\ 0, & \text{else}\end{cases}$.

$P(X_{n+1}=j, X_n=i, X_{n-1}=i_{n-1}, \cdots, X_0=i_0)\neq 0$ iff "$i_1-i_0, i_2-i_1, \cdots, i_{n-1}-i_{n-2}, i-i_{n-1}, j-i = 0$ or 1 and $i_0\geq 1$"

$\Rightarrow P(X_{n+1}=j\,|\,X_n=i, X_{n-1}=i_{n-1}, \cdots, X_0=i_0)=0$ when $j\neq i, i+1$.

$P(X_{n+1}=i\,|\,X_n=i, X_{n-1}=i_{n-1}, \cdots, X_0=i_0) = P(X_{n+1}=i, X_n=i, \cdots, X_0=i_0)\big/P(X_n=i, \cdots, X_0=i_0)$

$= P(\xi_{n+1}\in\{\xi_1,\cdots,\xi_n\}, X_n=i, \cdots, X_0=i_0)\big/P(X_n=i,\cdots,X_0=i_0) = \frac{i}{N}$

And similarly, $P(X_{n+1}=i+1\,|\,X_n=i,\cdots,X_0=i_0)=\frac{N-i}{N}$

Thus, $P(X_{n+1}=i+1\,|\,X_n=i,\cdots,X_0=i_0)=P(X_{n+1}=j\,|\,X_n=i)$ whenever the conditional probability makes sense.

And the transition probability $p(i,j)=\begin{cases} i/N, & j=i \\ 1-i/N, & j=i+1 \\ 0, & \text{else}\end{cases}$

5.1.2 Proof: $P(X_7=3\,|\,X_6=2, X_5=1)=\dfrac{P(X_7=3, X_6=2, X_5=1)}{P(X_6=2, X_5=1)}=\dfrac{P(\xi_7=1)\,P(X_6=2, X_5=1)}{P(X_6=2, X_5=1)}=\frac{1}{2}$

$P(X_7=3\,|\,X_6=2, X_5=2)=P(X_7=3\,|\,X_6=2, X_5=2, S_6=2)P(S_6=2)+P(X_7=3\,|\,X_6=2, X_5=2, S_6\leq 0)\,P(S_6\leq 0)$

$=\frac{1}{2}P(S_6=2)<\frac{1}{2}$

$\Rightarrow X_n$ is not a Markov chain.

5.1.3 Proof: $P(X_{n+1}=(i,j)\,|\,X_n=(k_n,k_{n+1}), X_{n-1}=(k_{n-1},k_n), \cdots, X_0=(k_0,k_1))$

$=P(X_{n+1}=(i,j), \xi_0=k_0, \xi_1=k_1, \cdots, \xi_{n+1}=k_{n+1})\big/P(\xi_0=k_0, \xi_1=k_1, \cdots, \xi_{n+1}=k_{n+1})$

$=\begin{cases} 0, & i\neq k_{n+1} \\ \frac{1}{2}, & i=k_{n+1}\end{cases}$ independent of $k_0, k_1, \cdots, k_n$.

$\Rightarrow X_n=(\xi_n,\xi_{n+1})$ is a Markov chain.

X_{n+1} \ X_n	(H,H)	(H,T)	(T,H)	(T,T)
(H,H)	$\frac{1}{2}$	0	$\frac{1}{2}$	0
(H,T)	$\frac{1}{2}$	0	$\frac{1}{2}$	0
(T,H)	0	$\frac{1}{2}$	0	$\frac{1}{2}$
(T,T)	0	$\frac{1}{2}$	0	$\frac{1}{2}$

$p^2(m,n)=\frac{1}{4}$ for any state m,n due to the independence of $\xi_n, \xi_{n+1}, \xi_{n+2}$.

Remark on Ex 5.2.2. Denote $\tilde{B}_n = \bigcup_{m=n+1}^{\infty} \{X_m \in B_m\} \in \mathcal{F}_\infty$. $P(\tilde{B}_n | \mathcal{F}_k) \to 1_{\tilde{B}_n}$ a.s. as $k \to \infty$ (Lévy's 0-1 law)

Suppose $k > n$. $P(1_{\tilde{B}_n} | \mathcal{F}_k) \geq P(1_{\tilde{B}_k} | \mathcal{F}_k) \geq \delta > 0$ on $\{X_n \in A_n\}$

$\Rightarrow 1_{\tilde{B}_n} \overset{a.s.}{=} \varliminf_{k\to\infty} P(\tilde{B}_n | \mathcal{F}_k) \geq \delta > 0$ on $\{X_n \in A_n \text{ i.o.}\}$ for any n.

$\Rightarrow P(\{X_n \in A_n \text{ i.o.}\} - \{X_n \in B_n \text{ i.o.}\}) = 0$

5.2.1 Proof: $B \in \sigma(X_n, X_{n+1}, \cdots) \Rightarrow \exists\, Y \in \mathcal{F}_0$, s.t. $1_B = Y \circ \theta_n$

$$P_\mu(A \cap B | X_n) = E_\mu(E_\mu(1_A 1_B | \mathcal{F}_n) | X_n) = E_\mu(1_A E_\mu(1_B | \mathcal{F}_n) | X_n)$$
$$= E_\mu(1_A E_\mu(Y \circ \theta_n | \mathcal{F}_n) | X_n) \overset{\text{Markov}}{=} E_\mu(1_A E_{X_n} Y | X_n)$$
$$= E_\mu(1_A | X_n) E_{X_n} Y = E_\mu(1_A | X_n) E_\mu(1_B | \mathcal{F}_n)$$

$1_B \in \sigma(X_n, X_{n+1}, \cdots) \Rightarrow E_\mu(1_B | \mathcal{F}_n) \in \mathcal{F}_n \cap \sigma(X_n, X_{n+1}, \cdots) = \sigma(X_n)$

$\Rightarrow E_\mu(1_B | \mathcal{F}_n) = E_\mu(E_\mu(1_B | \mathcal{F}_n) | X_n) = E_\mu(1_B | X_n)$

i.e. $P_\mu(A \cap B | X_n) = P_\mu(A | X_n) P_\mu(B | X_n)$

5.2.2 Proof: Let $B = \{X_n \in B_n \text{ i.o.}\}$, $\bar{B}_n = \bigcup_{m=n+1}^{\infty} \{X_m \in B_m\}$, then $B = \bigcap_{n=1}^{\infty} \bar{B}_n$, $\bar{B}_n \downarrow B$

$1_{\bar{B}_n} \to 1_B$ a.s. $|1_{\bar{B}_n}| \leq 1$ bounded.

And $P(\bigcup_{m=n+1}^{\infty} \{X_m \in B_m\} | X_n) = E(1_{\bar{B}_n} | X_n) = E(E(1_{\bar{B}_n} | \mathcal{F}_n) | X_n)$

$$= E(E_{X_n}(1_{\bigcup_{m=1}^{\infty}\{X_m \in B_{m+n}\}})) = E_{X_n} 1_{\bigcup_{m=1}^{\infty}\{X_m \in B_{m+n}\}} = E(1_{\bar{B}_n} | \mathcal{F}_n)$$

$\Rightarrow E(1_{\bar{B}_n} | \mathcal{F}_n) \geq \delta > 0$ on $\{X_n \in A_n\}$

$\mathcal{F}_n \uparrow \mathcal{F}_\infty$. By Dominated Convergence for conditional expectation, (Thm 4.6.10)

$E(1_{\bar{B}_n} | \mathcal{F}_n) \to E(1_B | \mathcal{F}_\infty) = 1_B$ a.s. Actually, Lévy's 0-1 law is its corrollary.

On $\{X_n \in A_n \text{ i.o.}\}$, $1_B = \varlimsup_{n\to\infty} E(1_{\bar{B}_n} | \mathcal{F}_n) \geq \delta > 0 \Rightarrow P(\{X_n \in A_n \text{ i.o.}\} - \{X_n \in B_n \text{ i.o.}\}) = 0$

5.2.3 Proof: Note that $D \in \mathcal{T}$, the tail σ-field, and $P_a(X_1 = a) = 1$.

$1_D \circ \theta_n = 1_{\{X_m = a \text{ for some } m \geq n\}} = 1_D$, P_μ-a.s.

$h(X_n) = E_{X_n} 1_D = E_\mu(1_D \circ \theta_n | \mathcal{F}_n) = E_\mu(1_D | \mathcal{F}_n) \to 1_D$ a.s. by Lévy's 0-1 law.

$\Rightarrow h(X_n) \to 0$ a.s. on D^c.

5.2.4 Proof: $p^n(x,y) = P(X_n = y | X_0 = x) = \sum_{m=1}^{n} P(X_n = y | T_y = m, X_0 = x) P(T_y = m | X_0 = x)$

By Markov property, $P(X_n = y | T_y = m, X_0 = x) = P(X_n = y | X_m = y, X_{m-1} \neq y, X_{m-2} \neq y, \cdots, X_0 = x)$

$$= P(X_n = y | X_m = y) = p^{n-m}(y,y) \text{ for } n \geq m.$$

$P(T_y = m | X_0 = x) = P_x(T_y = m)$, then $p^n(x,y) = \sum_{m=1}^{n} P_x(T_y = m) p^{n-m}(y,y)$

5.2.5 Proof: $T = \inf\{m \geq k : X_m = x\}$, $\forall m \geq k$

$$p^m(x,x) = P_x(X_m = x | X_0 = x) = \sum_{l=k}^{m} P_x(X_m = x | T = l, X_0 = x) P_x(T = l | X_0 = x)$$
$$= \sum_{l=k}^{m} P_x(X_m = x | X_l = x) P_x(T = l | X_0 = x) = \sum_{l=k}^{m} P_x(T = l) p^{m-l}(x,x)$$

$$\sum_{m=k}^{n+k} P_x(X_m = x) = \sum_{m=k}^{n+k} p^m(x,x) = \sum_{m=k}^{n+k} \sum_{l=k}^{m} p^{m-l}(x,x) P_x(T = l) = \sum_{m=k}^{n+k} \sum_{s=0}^{m-k} p^s(x,x) P_x(T = m-s)$$
$$= \sum_{s=0}^{n} p^s(x,x) \sum_{m=s+k}^{n+k} P_x(T = m-s) = \sum_{s=0}^{n} p^s(x,x) P_x(k \leq T \leq n+k-s) \leq \sum_{s=0}^{n} p^s(x,x) = \sum_{m=0}^{n} P_x(X_m = x)$$

5.2.6 Proof: $\forall x\in S-C$, $P_x(T_C<\infty)>0 \Rightarrow \exists N_x\in\mathbb{N}$, s.t $P_x(T_C\le N_x)>0$

Denote $N=\max_{x\in S-C} N_x$, then $N<\infty$ since $S-C$ is finite

$\varepsilon=\min_{x\in S-C} P_x(T_C\le N)>0$

When $k=1$, $P_y(T_C>N)=1-P_y(T_C\le N)\le 1-\varepsilon$, $\forall y\in S-C$

Suppose $P_y(T_C>(k-1)N)\le(1-\varepsilon)^{k-1}$, $\forall y\in S-C$

$P_y(T_C>kN)=P_y(T_C>kN\mid T_C>(k-1)N)\,P_y(T_C>(k-1)N)$

$P_y(T_C>kN\mid T_C>(k-1)N)=P_y(X_n\notin C \text{ for } (k-1)N<n\le kN \mid X_n\notin C \text{ for } 1\le n\le(k-1)N)$

$=P_y(X_n\notin C \text{ for } (k-1)N<n\le kN\mid X_{(k-1)N}\notin C)$

$=\sum_{x\in S-C} P_y(X_n\notin C \text{ for } (k-1)N<n\le kN\mid X_{(k-1)N}=x)\,P_y(X_{(k-1)N}=x\mid X_{(k-1)N}\notin C)$

$=\sum_{x\in S-C} P_x(T_C>N)\,P_y(X_{(k-1)N}=x\mid X_{(k-1)N}\notin C)\le 1-\varepsilon$

$\Rightarrow P_y(T_C>kN)\le(1-\varepsilon)^k$, $\forall k\in\mathbb{N}$, $y\in S-C$

5.2.7 Proof: (i) It's trivial to verify that $1_{(V_A<V_B)}\circ\theta_1\equiv 1_{(V_A<V_B)}$ in Ω_0.

$h(x)=P_x(V_A<V_B)=E(E(1_{(V_A<V_B)}\mid\mathcal{F}_1))=E(E(1_{(V_A<V_B)}\circ\theta_1\mid\mathcal{F}_1))$

$\overset{\text{Markov}}{=} E(E_{X_1}1_{(V_A<V_B)})=\sum_y P_x(X_1=y)\,P_y(V_A<V_B)=\sum_y p(x,y)h(y)$

(ii) $h(X(n\wedge V_{A\cup B}))=h(X_n)1_{(n<V_{A\cup B})}+h(X_V)1_{(n\ge V_{A\cup B})}$

$E(h(X_{V\wedge(n+1)})1_{(n<V)}\mid\mathcal{F}_n)=E(h(X_{n+1})1_{(n<V)}\mid\mathcal{F}_n)=1_{(n<V)}E(h(X_{n+1})\mid\mathcal{F}_n)$

$=1_{(n<V)}E_{X_n}h(X_1)\overset{(i)}{=}1_{(n<V)}h(X_n)$

$E(h(X_{V\wedge(n+1)})1_{(n\ge V)}\mid\mathcal{F}_n)=E(h(X_V)1_{(n\ge V)}\mid\mathcal{F}_n)=h(X_V)1_{(n\ge V)}$

$\Rightarrow E(h(X_{V\wedge(n+1)})\mid\mathcal{F}_n)=h(X_{V\wedge n})$

i.e. $h(X(n\wedge V_{A\cup B}))$ is a martingale.

(iii) Suppose $g:S\to\mathbb{R}$ also solves the equation. By Ex 5.2.6., We have $V_{A\cup B}<\infty$ a.s.

$g\equiv1$ on A, $g\equiv0$ on B, $S-(A\cup B)$ is finite

$\Rightarrow g(X_{n\wedge V})$ is a bounded martingale

$g(x)=E_x g(X_{n\wedge V})\to E_x g(X_V)=P_x(V_A<V_B)$

5.2.8 Proof: Using Exercise 5.2.7, we conclude that $P_x(V_0\wedge V_N<\infty)=1$, $\forall x\in S$

By the convergence of the martingale $X(n\wedge V_0\wedge V_N)$, we have

$x=E_x X(0\wedge V_0\wedge V_N)=E_x X(V_0\wedge V_N)=N\,P(V_N<V_0)\Rightarrow P_x(V_N<V_0)=\frac{x}{N}$

5.2.11 Proof: (i) $\forall x\notin A$, $1+V_A\circ\theta_1 = V_A$ P_x-a.s.

$$g(x) = E_x V_A = E_x E_x(V_A|\mathcal{F}_1) = E_x E_x(V_A\circ\theta_1|\mathcal{F}_1)+1$$

$$\overset{\text{Markov}}{=\!=} E_x E_{X_1}(V_A)+1 = \sum_y p(x,y)\,g(y)+1$$

(ii) $$E(g(X_{(n+1)\wedge V})|\mathcal{F}_n) = E(g(X_{n+1})|\mathcal{F}_n)1_{(V\ge n+1)} + g(X_V)1_{(V\le n)}$$

$$= E(g(X_1)\circ\theta_n|\mathcal{F}_n)1_{(V\ge n+1)} + g(X_V)1_{(V\le n)}$$

$$= E_{X_n} g(X_1)\cdot 1_{(V\ge n+1)} + g(X_V)1_{(V\le n)}$$

$$= \Big(\sum_y p(X_n,y)\,g(y)\Big)1_{(V\ge n+1)} + g(X_V)1_{(V\le n)}$$

$$= g(X_n)1_{(V\ge n+1)} + g(X_V)1_{(V\le n)} - 1_{(V\ge n+1)} = g(X_{n\wedge V}) - 1_{(V\ge n+1)}$$

(Note that $V\ge n+1$ implies $X_n\notin A$.)

$\Rightarrow X_{n\wedge V_A} + n\wedge V_A$ is a martingale.

(iii) Employing Exercise 5.2.6, $P_x(V_A<\infty)=1$ for $\forall x\in S$

By convergence of the martingale in (ii)

$$g(x) = E_x(g(X_0)+0) = E_x(g(X_{V_A})+V_A) \xrightarrow[g(X_{V_A})=0]{X_{V_A}\in A} E_x V_A$$

5.3.1 Proof: Denote $V=\{(r,s_1,s_2,\cdots,s_r)\mid r\ge 1,\ s_i\in S,\ 1\le i\le r\}$ to be all the possible values of the vectors. Remark: 设S为 state space. 则 $V=\bigsqcup_{k\ge1} S^k$ (即V为卡氏积S^k的无交并)

V is countable, then the σ-field on V is 2^V. $V_k:\Omega_0\to V$ are random elements. 这样V上就有自然的σ代数.

y is recurrent $\Rightarrow P_y(R_k<\infty)=1$. $\forall k\ge 1$. 题设中的V_k就是良定义的 random element.

$V_k = V_1\circ\theta_{R_k}$. $\forall v,\tilde v\in V$, $k>m$.

$$P_y(V_k=v,\ V_m=\tilde v) = E_y(1_{(V_k=v)};\{V_m=\tilde v\}) = E_y(E_y(1_{(V_k=v)}|\mathcal{F}_{R_k});\{V_m=\tilde v\})$$

$(\{V_m=\tilde v\}\in\mathcal{F}_{R_k})$

$$\overset{\text{Markov}}{=\!=} E_y(E_y(1_{(V_1=v)}\circ\theta_{R_k}|\mathcal{F}_{R_k});\{V_m=\tilde v\})$$

$$= E_y(P_{X_{R_k}}(V_1=v);\{V_m=\tilde v\}) = P_y(V_1=v)\,P_y(V_m=\tilde v) \quad (\text{Note that } X_{R_k}=y)$$

So it suffices to prove that $P_y(V_1=v)=P_y(V_k=v)$ to show that V_k, V_m are independent.

It's trivial to see that

$$P_y(V_k=v) = E_y E_y(1_{(V_1=v)}\circ\theta_{R_k}|\mathcal{F}_{R_k}) = E_y E_{X_{R_k}}(1_{(V_1=v)}) = P_y(V_1=v)$$

This also verifies that V_k's are identically distributed.

In general, $\{V_k\}_{k\ge1}$ is i.i.d.

5.3.2 Proof: Firstly verify that $1_{(T_z<\infty)}\circ\theta_{T_y} \le 1_{(T_z<\infty)}$ on $\{T_y<\infty\}$ P_x-a.s.

Under the very circumstances that the path starts from x, visits z,y in turn and never revisits z afterwards, will the strict inequality holds. Otherwise, the equation holds.

$$\rho_{xz} = P_x(T_z<\infty) \ge E_x(1_{(T_z<\infty)}\circ\theta_{T_y} ; T_y<\infty)$$
$$= E_x(E_x(1_{(T_z<\infty)}\circ\theta_{T_y} | \mathcal{F}_{T_y}) ; T_y<\infty) = E_x(E_{X_{T_y}} 1_{(T_z<\infty)} ; T_y<\infty)$$
$$= E_x(\rho_{yz} ; T_y<\infty) = \rho_{xy}\rho_{yz}.$$

5.3.3 Proof: S is closed and finite.

$\forall\, i\ne j \in S$.

If $i<j$, then $\rho_{ij} = P_i(T_j<\infty) \ge p^{j-i}(i,j) \ge p(i,i+1)\,p(i+1,i+2)\cdots p(j-1,j) > 0$

If $i>j$, then $\rho_{ij} = P_i(T_j<\infty) \ge p^{i-j}(i,j) \ge p(i,i-1)\,p(i-1,i-2)\cdots p(j+1,j) > 0$

$\Rightarrow S$ is irreduccible.

Applying Thm 5.3.3, we conclude that all states in S are recurrent.

5.3.7 Proof: Provided that p is recurrent.

$\forall x,y\in S$, $\rho_{xy} = P_x(T_y<\infty) > 0$ due to the irreducibility of p.

Applying Thm 5.3.2, we conclude $\rho_{xy}=1$ because y is recurrent.

$f(X_{n\wedge T_y})$ is a nonnegative supermartingale and converges P_x-a.s. to $f(X_{T_y})$

$\Rightarrow f(x) = E_x f(X_{0\wedge T_y}) \ge E_x f(X_{T_y}) = f(y)$

Similarly, we can prove that $f(x)\le f(y)$. Henceforth, f is constant.

If p is transient, then all states are transient considering y is irreducible.

$\exists\, x,y,z\in S$, s.t. $P_x(T_y<\infty) \ne P_z(T_y<\infty)$

Keeping y fixed, assign $f(x) = P_x(T_y<\infty)$.

Then f is a nonconstant nonnegative superharmonic function.

5.5.2 Proof: $\mu_x(y)\,w_{yx} = \sum_{n=0}^{\infty} P_x(X_n=y, T_x>n)\,P_y(T_x<T_y) = \sum_{n=0}^{\infty} E_x(P_{X_n}(T_x<T_y) ; X_n=y, T_x>n)$

(文档末有更好的证明)

$$= \sum_{n=0}^{\infty} E_x(E_x(1_{(T_x<T_y)}\circ\theta_n | \mathcal{F}_n) ; X_n=y, T_x>n)$$
$$= \sum_{n=0}^{\infty} E_x(1_{(T_x<T_y)}\circ\theta_n ; X_n=y, T_x>n)$$

Denote $A_n = \{1_{(T_x<T_y)}\circ\theta_n = 1, X_n=y, T_x>n\} \subseteq \{T_y<T_x\}$

$n<m$. Suppose $\omega\in A_n\cap A_m$, then $n<m<T_x$, contradict to $1_{(T_x<T_y)}\circ\theta_n = 1$.

i.e. A_n's are disjoint. Moreover, if $\omega\in\{T_y<T_x\}$,

Set $N(\omega) = \max\{1\le n\le T_x \mid X_n=y\}$. then $\omega\in A_{N(\omega)}$. In general, $\{T_y<T_x\} = \bigsqcup_{n=1}^{\infty} A_n$

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$w_{yx}\,\mu_x(y) = \sum_{n=0}^{\infty} P_x(A_n) = P_x(T_y < T_x) = w_{xy}$ $\quad (P_x(A_0)=0)$

i.e. $\mu_x(y) = \dfrac{w_{xy}}{w_{yx}}$

5.5.3 Proof: x,y fixed, then $\mu_x(z), \mu_y(z)$ are both stationary measures on S.

Applying Thm 5.5.9, $\dfrac{\mu_x(z)}{\mu_y(z)}$ is a constant invariant of z.

Choosing $z=y$, we conclude that $\mu_x(z) = \dfrac{\mu_x(y)}{\mu_y(y)}\mu_y(z) = \mu_x(y)\mu_y(z)$

5.5.4 Proof: Set $A=\{T_x<T_y\}$. $\forall n\ge 1$

文档末有8证 $A\cap\{T_x=n\} = \{X_i\ne x,y,\ 1\le i\le n-1,\ X_n=x\}\in\mathcal{F}_n \Rightarrow A\in\mathcal{F}_{T_x}$

$P_y(A) = w_{yx} > 0$ because p is positive reccurent. And $T_y\circ\theta_{T_x} = T_y - T_x$ on $\{T_x<T_y\}$

$$P_y(A)E_xT_y = E_y(E_xT_y\,;T_x<T_y) = E_y(E_{X_{T_x}}T_y\,;T_x<T_y) = E_y(E_y(T_y\circ\theta_{T_x}\mid\mathcal{F}_{T_x});T_x<T_y)$$
$$= E_y(T_y\circ\theta_{T_x}\,;T_x<T_y) = E_y(T_y-T_x\,;T_x<T_y) \le E_yT_y<\infty$$

$\Rightarrow E_xT_y<\infty$.

5.5.5 Proof: Suppose p is positive recurrent.

$\Rightarrow \exists x$, s.t. $E_xT_x<\infty$

Applying Thm 5.5.12 and Thm 5.5.15,

p only has finite stationary measure, contradiction!

5.5.6 Proof: ~~Considering the simple random walk~~, ~~$S=\mathbb{Z}$, $p(x,x+1)=p$, $p(x,x-1)=q=1-p$ $(0<p<1)$~~

The transition probability is obviously irreducible, thus it has one unique stationary measure up to a constant multiples.

And $\mu(x)\equiv 1$ $(\forall x\in\mathbb{Z})$ is a stationary measure in particular.

(i) $\mu_0(k) = \mu_0(0)\dfrac{\mu(k)}{\mu(0)} = 1$, $\forall k\in\mathbb{Z}$ $\quad$ (We have to assume the random walk is symmetric) ☆

i.e. $E_0\left(\sum_{n=0}^{T_0-1} 1_{(X_n=k)}\right) = 1$

(ii) Claim: $k\ge1$, then $E_1\sum_{n=0}^{T_0-1} 1_{\{X_n=k\}} = 2$

$$1 = \mu_0(k) = E_0\sum_{n=0}^{T_0-1}1_{\{X_n=k\}} = E_0\Big(\sum_{n=0}^{T_0-1}1_{\{X_n=k\}}\,;X_1=1\Big) + E_0\Big(\sum_{n=0}^{T_0-1}1_{\{X_n=k\}}\,;X_1=-1\Big)$$

(the last term $=0$ obviously and $\sum_{n=0}^{T_0-1}1_{\{X_n=k\}}\circ\theta_1 = \sum_{n=0}^{T_0-1}1_{\{X_n=k\}}$ on $\{X_1=1, X_0=0\}$)

$$= E_0\Big(\sum_{n=0}^{T_0-1}1_{\{X_n=k\}}\circ\theta_1\,;X_1=1\Big) = E_0\Big(E_0\Big(\sum_{n=0}^{T_0-1}1_{\{X_n=k\}}\circ\theta_1\mid\mathcal{F}_1\Big);X_1=1\Big)$$
$$= E_0\Big(E_1\sum_{n=0}^{T_0-1}1_{\{X_n=k\}}\,;X_1=1\Big) = \frac{1}{2}E_1\sum_{n=0}^{T_0-1}1_{\{X_n=k\}}, \text{ then the claim is proved.}$$

Assume $E_{k-1}\sum_{n=0}^{T_0-1}1_{\{X_n=k-1\}} = 2(k-1)$, we can get $E_k\sum_{n=0}^{T_1-1}1_{\{X_n=k\}} = 2(k-1)$ by translation. $\cdots(*)$

$$E_k\sum_{n=T_1}^{T_0-1}1_{\{X_n=k\}} = E_k\Big(\sum_{n=0}^{T_0-1}1_{\{X_n=k\}}\circ\theta_{T_1}\Big) = E_1\sum_{n=0}^{T_0-1}1_{\{X_n=k\}} = 2$$

Combining with $(*)$, we have $E_k\sum_{n=0}^{T_0-1}1_{\{X_n=k\}} = 2k$. $\Rightarrow$ The result is proved by induction.

5.6.1 Proof: When $n=0$, $P_\mu(X_0=0)=\mu(0)=\frac{\beta}{\alpha+\beta}+(1-\alpha-\beta)^0\left(\mu(0)-\frac{\beta}{\alpha+\beta}\right)$

Suppose when $n\geq 0$, $P_\mu(X_n=0)=\frac{\beta}{\alpha+\beta}+(1-\alpha-\beta)^n\left(\mu(0)-\frac{\beta}{\alpha+\beta}\right)$

$$\begin{aligned} P_\mu(X_{n+1}=0) &= P_\mu(X_n=0)\,p(0,0)+P_\mu(X_n=1)\,p(1,0) \\ &= P_\mu(X_n=0)\cdot(1-\alpha)+(1-P_\mu(X_n=0))\cdot\beta \\ &= \beta+P_\mu(X_n=0)(1-\alpha-\beta) \\ &= \frac{\beta}{\alpha+\beta}+(1-\alpha-\beta)^{n+1}\left(\mu(0)-\frac{\beta}{\alpha+\beta}\right) \end{aligned}$$

Following by induction, we conclude that the result is true for any $n\geq 1$.

5.6.2 Proof: Since p is aperiodic, $\exists M_x\in\mathbb{N}$ for any $x\in S$, s.t. $p^m(x,x)>0$ for any $m\geq M_x$.

$M=\sup_{x\in S} M_x<\infty$ because S is finite.

p is irreducible $\Rightarrow \forall x,y\in S$, $\exists m_{x,y}\in\mathbb{N}$, s.t. $p^{m_{x,y}}(x,y)>0$

$\tilde{M}:=M+\sup_{x,y\in S} m_{x,y}<\infty$ again by the assertion that S is finite.

$\forall m\geq\tilde{M}$, $\forall x,y\in S$, $p^m(x,y)\geq p^{m_{x,y}}(x,y)\,p^{m-m_{x,y}}(y,y)>0$ $(m-m_{x,y}\geq M\geq M_y)$

5.6.3 Proof: By Exercise 5.6.2, $\exists m\in\mathbb{N}$, s.t. $p^m(x,y)>0$ for all $x,y\in S$.

Denote $N=|S|$ to be the number of all states, $\varepsilon:=\inf_{x,y} p^m(x,y)>0$

$P(X_{n+m}=Y_{n+m}\mid X_n=x,Y_n=y)=\sum_z p^m(x,z)p^m(y,z)\geq\varepsilon^2 N$, for any $x,y\in S$.

$$\begin{aligned} \Rightarrow P(T>n+m\mid T>n) &= \sum_{x,y} P(X_{n+m}\neq Y_{n+m}\mid X_n=x,Y_n=y)\,P(X_n=x,Y_n=y\mid T>n) \\ &\leq \sum_{x,y}(1-\varepsilon^2 N)\,P(X_n=x,Y_n=y\mid T>n)=1-\varepsilon^2 N. \end{aligned}$$

$C:=\sup_{0\leq n<m} P(T>n)<\infty$.

Then we have $P(T>n)\leq C(1-\varepsilon^2 N)^{\frac{n}{m}}$.

Another proof of Ex 5.5.2

Denote $N(y)=\sum_{n=0}^{T_x-1} 1_{\{X_n=y\}}$, then $\mu_x(y)=E_x N(y)=\sum_{n=1}^{\infty} P_x(N(y)\geq n)$

Obviously, $P_x(N(y)\geq 1)=P_x(T_y<T_x)=w_{xy}$

And $P_x(N(y)\geq n) = E_x\left(1_{\{T_y^n<T_x\}};\, T_y^{n-1}<T_x\right) \xlongequal[1_{\{T_y^n<T_x\}}=1_{\{T_y<T_x\}}\circ\theta_{T_y^{n-1}}]{\{T_y^{n-1}<T_x\}\in\mathcal{F}_{T_y^{n-1}}} E_x\left(E_x(1_{\{T_y<T_x\}}\circ\theta_{T_y^{n-1}}\mid\mathcal{F}_{T_y^{n-1}});\, T_y^{n-1}<T_x\right)$

$\xlongequal{\text{Markov}} E_x\left(E_{X_{T_y^{n-1}}} 1_{\{T_y<T_x\}};\, T_y^{n-1}<T_x\right) = (1-w_{yx})P(N(y)\geq n-1)$

$\Rightarrow P_x(N(y)\geq n) = w_{xy}(1-w_{yx})^{n-1}$

Sum over n to prove that $\mu_x(y)=w_{xy}/w_{yx}$.

Another proof of Ex 5.5.4

$p(y,x)\,P_x(T_y\geq n) \xlongequal{\text{Markov}} P_y(T_y\geq n+1,\, X_1=x) \leq P_y(T_y\geq n+1)$

$\Rightarrow p(y,x)\,E_xT_y=\sum_{n=1}^{\infty} p(y,x)P_x(T_y\geq n) \leq \sum_{n=1}^{\infty} P_y(T_y\geq n+1) \leq E_yT_y<\infty$ by the positive recurrence.

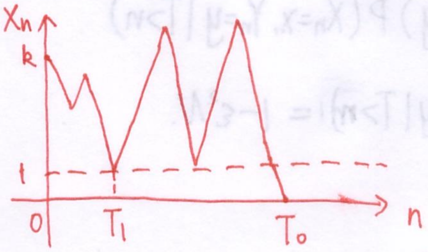
$p(y,x)>0$ due to the irreducibility

$\Rightarrow E_xT_y<\infty$.

Remark on Ex 5.5.6 (ii)

After verifying the claim that "$E_1\sum_{n=0}^{T_0-1} 1_{\{X_n=k\}}=2$ for any $k\geq 1$", we have two interpretations on how to prove the ultimate result.

1° Induction on k



在第一次到1之前，X_n的运动实质上同$k-1$的情形

即 $E_k\sum_{n=0}^{T_1-1} 1_{\{X_n=k\}}=2(k-1)$

(将 $k-1$ 时的结论平移可得)

从T_1到T_0之间运动的过程，就是claim的内容。

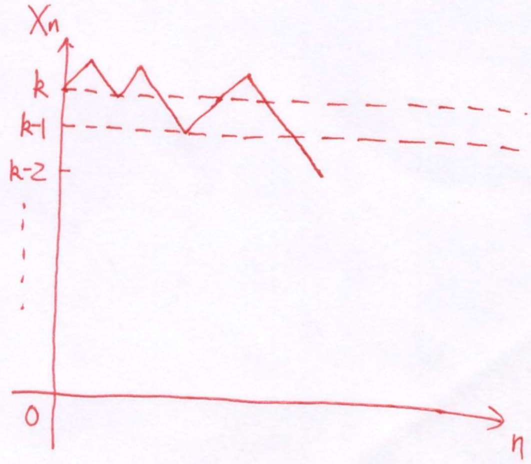
2° 将claim的结果平移得 $E_{l+1}\sum_{n=0}^{T_l-1} 1_{\{X_n=k\}}=2$ provided that $k\geq l$.

依次取 $l=k-1, k-2, \cdots, 0$ 得

从k到第一次到$k-1$，visits k 平均2次

从$k-1$到第一次到$k-2$，visits k 平均2次

⋮



再求和，就证完了。