

中国科学技术大学数学科学学院
2020—2021学年第二学期考试试卷

■ A 卷

□ B 卷

课程名称: 数学物理方程(A) 课程编号: 001506
姓名: 李治平 学号: PB1920649 专业: 物理

题号	一	二	三	四	五	六	七	八	总分
得分	11	16	5	13	10	13	10	6	75

一(12分) 求以下固有值问题的固有值,固有函数.

$$\begin{cases} y'' + \lambda y = 0, & (0 < x < 21) \\ y(0) = 0, & y(21) = 0. \end{cases}$$

二(16分) 考虑如下形式定解问题:

$$\begin{cases} u_{tt} = 4u_{xx} + f(t, x), & (t > 0, -\infty < x < +\infty) \\ u(0, x) = 2x^2, & u_t(0, x) = 3 \cos x \end{cases}$$

- (1) 在 $f(t, x) = 0$ 时, 求此问题的解.
(2) 在 $f(t, x) = t^2 x + 5 \sin 2t \sin x$ 时, 求此问题的解.

三(10分) 求解定解问题

$$\begin{cases} x \frac{\partial u}{\partial x} - 3y \frac{\partial u}{\partial y} + u = 0, \\ u(1, y) = y + 3y^2 \end{cases}$$

四.(14分) 求解混合问题:

$$\begin{cases} u_{tt} = 4u_{xx} + \delta(t-3, x-2), & (t > 0, 0 < x < 5) \\ u(t, 0) = u_x(t, 5) = 0, \\ u(0, x) = \sin \frac{1}{2} \pi x + 3 \sin \frac{3}{10} \pi x, & u_t(0, x) = 0. \end{cases}$$

五(12分) 利用分离变量求解以下边值问题:

$$\begin{cases} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2} = 0, & (r = \sqrt{x^2 + y^2} < 2, 0 < z < 3) \\ u|_{r=2} = 0, \\ u|_{z=0} = 8 - 2r^2, & u|_{z=3} = 0. \end{cases}$$

六. (16分) 已知以下形式初值问题

$$\begin{cases} u_t = 4u_{xx} + a^2u_{yy} + bu_x + cu, & (t > 0, -\infty < x, y < +\infty) \\ u|_{t=0} = \varphi(x) + g(x, y). \end{cases}$$

- (1) 当 $a = b = c = 0, g(x, y) = 0$ 时, 设 u 不依赖于 y , 求这时 $u = u(t, x)$ 的解的表达式。
 (2) 当 $a = 1, b = 3, c = 2, \varphi(x) = x, g(x, y) = 5e^{-x^2-y^2}$ 时, 求这时解 $u(t, x, y)$ 。

七. (10分) 求解以下定解问题, 其中 (r, θ, φ) 为球坐标.

$$\begin{cases} \Delta_3 u = 0, & 1 < r < 3 \\ u|_{r=1} = \cos 2\theta - 1 \\ u|_{r=3} = 21 + \cos \theta. \end{cases}$$

八(10分) 求解边值问题:
$$\begin{cases} u_{xx} + 4u_{yy} = 0, & (x > 3, y > 0) \\ u = \begin{cases} g(y), & \text{当 } x = 3, y > 0 \\ f(x), & \text{当 } x \geq 3, y = 0 \end{cases} \end{cases}$$

参考公式

1) 直角坐标系: $\Delta_3 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$, 柱坐标系: $\Delta_3 u = \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$,

球坐标系: $\Delta_3 u = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial u}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial u}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2}$.

2) Bessel函数在三类边界条件下的模平方分别为: $N_{\nu 1n}^2 = \frac{a^2}{2} J_{\nu+1}^2(\omega_{1n}a)$,

$N_{\nu 2n}^2 = \frac{1}{2} [a^2 - \frac{\nu^2}{\omega_{2n}^2}] J_{\nu}^2(\omega_{2n}a)$, $N_{\nu 3n}^2 = \frac{1}{2} [a^2 - \frac{\nu^2}{\omega_{3n}^2} + \frac{a^2 \alpha^2}{\beta^2 \omega_{3n}^2}] J_{\nu}^2(\omega_{3n}a)$.

Bessel函数有递推公式: $(x^{\nu} J_{\nu})' = x^{\nu} J_{\nu-1}$, $(x^{-\nu} J_{\nu})' = -x^{-\nu} J_{\nu+1}$,
 $2J_{\nu}' = J_{\nu-1} - J_{\nu+1}$, $2\nu x^{-1} J_{\nu} = J_{\nu-1} + J_{\nu+1}$.

3) n 阶Legendre多项式: $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$, $n = 0, 1, 2, 3, \dots$; 递推公式:

1. $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$, 2. $nP_n(x) - xP_n'(x) + P_{n-1}'(x) = 0$,

3. $nP_{n-1}(x) - P_n'(x) + xP_{n-1}'(x) = 0$, 4. $P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x)$.

4) $\frac{1}{\pi} \int_0^{+\infty} e^{-a^2 \lambda^2 t} \cos \lambda x d\lambda = \frac{1}{2a\sqrt{\pi t}} \exp(-\frac{x^2}{4a^2 t})$

5) 由Poisson方程第一边值问题的格林函数 $G(M; M_0)$, 求相应问题的解 $u(M)$ 的公式:

空间区域: $u(M) = \iiint_V f(M_0) G(M; M_0) dM_0 - \iint_S \varphi(M_0) \frac{\partial G}{\partial n_0}(M; M_0) dS_0$,

平面区域: $u(M) = \iint_D f(M_0) G(M; M_0) dM_0 - \int_l \varphi(M_0) \frac{\partial G}{\partial n_0}(M; M_0) dl_0$.

参考解答和参考评分标准

一 解: 由S-L定理, 可设 $\lambda = \omega^2 > 0$, 这样解得

$$y = A \cos \omega x + B \sin \omega x, \dots\dots\dots 6 \text{ 分}$$

由 $y(0) = 0 \implies A = 0$, $y(21) = B \sin 21\omega = 0$, 因此 $21\omega = n\pi \implies \omega = \frac{n\pi}{21}$, $n = 1, 2, 3, \dots$ 得

$$\text{固有值: } \lambda_n = \left(\frac{n\pi}{21}\right)^2, \text{ 固有函数: } y_n(x) = \sin \frac{n\pi x}{21} \dots\dots\dots 12 \text{ 分}$$

二 解: (1) 在 $f(t, x) = 0$ 时, 由达朗贝尔公式:

$$u = \frac{2}{2} [(x+2t)^2 + (x-2t)^2] + \frac{1}{2 \times 2} \int_{x-2t}^{x+2t} 3 \cos \xi d\xi = 2x^2 + 8t^2 + \frac{3}{2} \sin 2t \cos x$$

$\dots\dots\dots 8 \text{ 分}$

(2) 在 $f(t, x) = t^2 x + 5 \sin 2t \sin x$ 时, 由叠加原理: $u = u_1 + u_2 + u_3$, 其中

$$\begin{cases} u_{1tt} = 4u_{1xx}, & (t > 0, -\infty < x < +\infty) \\ u_1(0, x) = 2x^2, & u_{1t}(0, x) = 3 \cos x \end{cases}$$

$$\begin{cases} u_{2tt} = 4u_{2xx} + t^2 x, & (t > 0, -\infty < x < +\infty) \\ u_2(0, x) = 0, & u_{2t}(0, x) = 0 \end{cases}$$

和

$$\begin{cases} u_{3tt} = 4u_{3xx} + 5 \sin 2t \sin x, & (t > 0, -\infty < x < +\infty) \\ u_3(0, x) = 0, & u_{3t}(0, x) = 0 \end{cases}$$

由第一问的结论: $u_1 = 2x^2 + 8t^2 + \frac{3}{2} \sin 2t \cos x$, 直接观察可得: $u_2 = \frac{t^4}{12} x$, 使用冲量原理, 求得:

$$\begin{aligned} u_3 &= \frac{5}{2 \times 2} \int_0^t d\tau \int_{x-2(t-\tau)}^{x+2(t-\tau)} \sin 2\tau \sin \xi d\xi = \frac{5}{2} \int_0^t \sin x \sin 2(t-\tau) \sin 2\tau d\tau \\ &= \frac{5}{4} \sin x \int_0^t (\cos(2t-4\tau) - \cos 2t) d\tau = \frac{5}{8} \sin 2t \sin x - \frac{5}{4} t \cos 2t \sin x \end{aligned}$$

综上, 这时

$$u = 2x^2 + 8t^2 + \frac{3}{2} \sin 2t \cos x + \frac{t^4}{12} x + \frac{5}{8} \sin 2t \sin x - \frac{5}{4} t \cos 2t \sin x$$

$\dots\dots\dots 16 \text{ 分}$

三 解: 特征方程为:

$$\frac{dx}{x} = \frac{dy}{-3y}$$

得到首次积分: $x^3 y = c$, 因此作变换:

$$\xi = x^3 y, \quad \eta = y, \quad \dots\dots\dots 5 \text{分}$$

方程化为:

$$-3\eta \frac{\partial u}{\partial \eta} + u = 0. \implies u = f(\xi)(\eta)^{\frac{1}{3}} = f(x^3 y)y^{\frac{1}{3}}$$

再利用定解条件 $u(1, y) = y + 3y^2$, 得到 $f(\xi) = \xi^{\frac{5}{3}} + \xi^{\frac{2}{3}}$, 最后得到

$$u(x, y) = 3x^5 y^2 + x^2 y$$

$\dots\dots\dots 10 \text{分}$

四.. 解: 使用叠加原理: $u = u_1 + u_2$ 其中

$$\begin{cases} u_{1tt} = 4u_{1xx}, & (t > 0, \quad 0 < x < 5) \\ u_1(t, 0) = u_{1x}(t, 5) = 0, \\ u_1(0, x) = \sin \frac{1}{2}\pi x + 3 \sin \frac{3}{10}\pi x, & u_{1t}(0, x) = 0. \end{cases}$$

以及

$$\begin{cases} u_{2tt} = 4u_{2xx} + \delta(t - 3, x - 2), & (t > 0, \quad 0 < x < 5) \\ u_2(t, 0) = u_{2x}(t, 5) = 0, \\ u_2(0, x) = 0, & u_{2t}(0, x) = 0. \end{cases}$$

为了求 u_1 , 作分离变量 $u_1 = T(t)X(x)$, 得到固有值问题:

$$\begin{cases} X'' + \lambda X = 0, & (0 < x < 5) \\ X(0) = 0, & X'(5) = 0. \end{cases}$$

和常微: $T'' + 4\lambda T = 0$. 其中固有值问题的固有值和固有函数为:

$$\lambda_n = \left(\frac{2n\pi + \pi}{10} \right)^2, \quad X_n(x) = \sin \frac{2n\pi + \pi}{10} x$$

相应地 $T_n(t) = C_n \cos \frac{2n\pi + \pi}{5} t + D_n \sin \frac{2n\pi + \pi}{5} t$, 这样, 由叠加原理可设:

$$u_1(t, x) = \sum_{n=0}^{+\infty} \left(C_n \cos \frac{2n\pi + \pi}{5} t + D_n \sin \frac{2n\pi + \pi}{5} t \right) \sin \frac{2n\pi + \pi}{10} x$$

再由 u_1 的初值条件, 定出:

$$u_1 = 3 \cos \frac{3\pi}{5} t \sin \frac{3}{10} \pi x + \cos \pi t \sin \frac{1}{2} \pi x$$

$\dots\dots\dots 6 \text{分}$

为了求 u_2 , 利用冲量原理

$$u_2 = \int_0^t W(t, x, \tau) d\tau$$

$W(t, x, \tau)$ 满足:

$$\begin{cases} W_{tt} = 4W_{xx}, & (t > \tau, \quad 0 < x < 5) \\ W(t, 0) = W_x(t, 5) = 0, \\ W|_{t=\tau} = 0, \quad W_t|_{t=\tau} = \delta(\tau - 3, x - 2). \end{cases}$$

利用变换 $t_1 = t - \tau$, 并利用分离变量法, 类似求得:

$$W(t, x, \tau) = \sum_{n=0}^{+\infty} \left(F_n \cos \frac{2n\pi + \pi}{5}(t - \tau) + G_n \sin \frac{2n\pi + \pi}{5}(t - \tau) \right) \sin \frac{2n\pi + \pi}{10}x$$

再利用 W 在 $t = \tau$ 的条件, 得到 $F_n = 0$, 而 G_n 满足:

$$\sum_{n=0}^{+\infty} G_n \frac{2n\pi + \pi}{5} \sin \frac{2n\pi + \pi}{10}x = \delta(\tau - 3, x - 2) = \delta(\tau - 3)\delta(x - 2).$$

得出:

$$G_n = \frac{2}{2n\pi + \pi} \sin \frac{2n\pi + \pi}{5} \delta(\tau - 3)$$

即

$$W(t, x, \tau) = \sum_{n=0}^{+\infty} \left(\frac{2}{2n\pi + \pi} \delta(\tau - 3) \sin \frac{2n\pi + \pi}{5} \sin \frac{2n\pi + \pi}{5}(t - \tau) \right) \sin \frac{2n\pi + \pi}{10}x$$

因此

$$u_2(t, x) = \begin{cases} \sum_{n=0}^{+\infty} \left(\frac{2}{2n\pi + \pi} \sin \frac{2n\pi + \pi}{5} \sin \frac{2n\pi + \pi}{5}(t - 3) \right) \sin \frac{2n\pi + \pi}{10}x, & t \geq 3 \\ 0, & t < 3 \end{cases}$$

综上, 此定解问题的解:

$u(t, x) =$

$$\begin{cases} 3 \cos \frac{3\pi}{5}t \sin \frac{3}{10}\pi x + \cos \pi t \sin \frac{1}{2}\pi x + \sum_{n=0}^{+\infty} \left[\frac{2}{2n\pi + \pi} \sin \frac{2n\pi + \pi}{5} \sin \frac{2n\pi + \pi}{5}(t - 3) \right] \sin \frac{2n\pi + \pi}{10}x, & t \geq 3 \\ 3 \cos \frac{3\pi}{5}t \sin \frac{3}{10}\pi x + \cos \pi t \sin \frac{1}{2}\pi x, & t < 3. \end{cases}$$

..... 14分

五 解: 作分离变量, 令 $u = R(r)Z(z)$, 则有

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dR(r)}{dr} \right) + \frac{d^2 Z(z)}{dz^2} = 0$$

考虑到齐次边界条件并在 $r = 0$ 附加自然边界条件, 得到 Bessel 方程固有值问题:

$$\begin{cases} r^2 R'' + rR' + \lambda r^2 R = 0, \\ |R(0)| < +\infty, R(2) = 0 \end{cases}$$

和微分方程 $Z'' - \lambda Z = 0$. 进一步求得一系列分离变量形式的解:

$$u_n(r, z) = (A_n \operatorname{ch} \omega_n z + B_n \operatorname{sh} \omega_n z) J_0(\omega_n r)$$

其中 ω_n 是代数方程 $J_0(2\omega) = 0$ 的第 n 个正根。由叠加原理,

$$u(r, z) = \sum_{n=1}^{+\infty} (A_n \operatorname{ch} \omega_n z + B_n \operatorname{sh} \omega_n z) J_0(\omega_n r)$$

利用边界条件:

$$u(r, 3) = \sum_{n=1}^{+\infty} (A_n \operatorname{ch} 3\omega_n + B_n \operatorname{sh} 3\omega_n) J_0(\omega_n r) = 0 \implies B_n = -\frac{\operatorname{ch} 3\omega_n}{\operatorname{sh} 3\omega_n} A_n = -\operatorname{cth} 3\omega_n A_n$$

$$u(r, 0) = \sum_{n=1}^{+\infty} A_n J_0(\omega_n r) = 8 - r^2$$

..... 7分

结合递推公式, 可算出广义Fourier系数:

$$\begin{aligned} A_n &= \frac{\int_0^2 r(8 - 2r^2) J_0(\omega_n r) dr}{N_{0n}^2} = 2 \frac{\int_0^2 r(4 - r^2) J_0(\omega_n r) dr}{N_{0n}^2} = \frac{2}{N_{0n}^2} \frac{1}{\omega_n^2} \int_0^{2\omega_n} t \left(4 - \frac{t^2}{\omega_n^2} \right) J_0(t) dt \\ &= \frac{2}{N_{0n}^2 \omega_n^2} \left[\left(4 - \frac{t^2}{\omega_n^2} \right) t J_1(t) \Big|_0^{2\omega_n} + \frac{2}{\omega_n^2} \int_0^{2\omega_n} t^2 J_1(t) dt \right] = \frac{8 J_2(2\omega_n)}{\omega_n^2 J_1^2(2\omega_n)} = \frac{16}{\omega_n^3 J_1(2\omega_n)}. \end{aligned}$$

进一步

$$B_n = -\frac{16 \operatorname{cth} 3\omega_n}{\omega_n^3 J_1(2\omega_n)}$$

最后

$$u(r, z) = \sum_{n=1}^{+\infty} \left(\frac{16}{\omega_n^3 J_1(2\omega_n)} \operatorname{ch} \omega_n z - \frac{16 \operatorname{cth} 3\omega_n}{\omega_n^3 J_1(2\omega_n)} \operatorname{sh} \omega_n z \right) J_0(\omega_n r)$$

..... 12分

六. 解: (1) 这时定解问题对应于:

$$\begin{cases} u_t = 4u_{xx} & (t > 0, -\infty < x < +\infty) \\ u|_{t=0} = \varphi(x). \end{cases}$$

作Fourier变换, 令 $\bar{u}(t, \lambda) = \int_{-\infty}^{+\infty} u(t, x) e^{-i\lambda x} dx$, 则经过Fourier变换, 像函数 $\bar{u}$ 满足:

$$\begin{cases} \frac{d\bar{u}}{dt} = -4\lambda^2 \bar{u} \\ \bar{u}|_{t=0} = \bar{\varphi}(\lambda) \end{cases}$$

于是解得:

$$\bar{u} = \bar{\varphi}(\lambda) e^{-4\lambda^2 t}$$

作反变换:

$$u(t, x) = \frac{1}{4\sqrt{\pi t}} \exp\left(-\frac{x^2}{16t}\right) * \varphi(x) = \frac{1}{4\sqrt{\pi t}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(x-\xi)^2}{16t}\right) \varphi(\xi) d\xi$$

.....8分

(2) 当 $a = 1, b = 3, c = 2, \varphi(x) = x, g(x, y) = 5e^{-x^2-y^2}$ 时,

$$\begin{cases} u_t = 4u_{xx} + u_{yy} + 3u_x + 2u, & (t > 0, -\infty < x, y < +\infty) \\ u|_{t=0} = x + 5e^{-x^2-y^2}. \end{cases}$$

令 $\bar{u}(t, \lambda, \mu) = \iint_{-\infty}^{+\infty} u(t, x, y) e^{-i(\lambda x + \mu y)} dx dy$, 则经过 Fourier 变换, 像函数 $\bar{u}$ 满足:

$$\begin{cases} \frac{d\bar{u}}{dt} = (-4\lambda^2 - \mu^2 + 3i\lambda + 2)\bar{u} \\ \bar{u}|_{t=0} = 2\pi h(\lambda)\delta(u) + G(\lambda, \mu) \end{cases}$$

其中 $h(\lambda) = \int_{-\infty}^{+\infty} x e^{-i\lambda x} dx, G(\lambda, \mu) = \iint_{-\infty}^{+\infty} 5e^{-x^2-y^2} e^{-i(\lambda x + \mu y)} dx dy$, 于是解得:

$$\bar{u}(t, \lambda, \mu) = 2\pi h(\lambda)\delta(u) e^{(-4\lambda^2 - \mu^2 + 3i\lambda + 2)t} + G(\lambda, \mu) e^{(-4\lambda^2 - \mu^2 + 3i\lambda + 2)t}$$

作反变换:

$$\frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} 2\pi h(\lambda)\delta(u) e^{(-4\lambda^2 - \mu^2 + 3i\lambda + 2)t} e^{i(\lambda x + \mu y)} d\lambda d\mu = \frac{e^{2t}}{2\pi} \int_{-\infty}^{+\infty} h(\lambda) e^{-4\lambda^2 t} e^{i\lambda(x+3t)} d\lambda$$

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{+\infty} h(\lambda) e^{-4\lambda^2 t} e^{i\lambda x} d\lambda &= x * \frac{1}{4\sqrt{\pi t}} \exp\left(-\frac{x^2}{16t}\right) = \frac{1}{4\sqrt{\pi t}} \int_{-\infty}^{+\infty} (x-\xi) \exp\left(-\frac{\xi^2}{16t}\right) d\xi \\ &= \frac{x}{4\sqrt{\pi t}} \int_{-\infty}^{+\infty} \exp\left(-\frac{\xi^2}{16t}\right) d\xi = \frac{x}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\tau^2} d\tau = x \end{aligned}$$

这样

$$F^{-1} \left[2\pi h(\lambda)\delta(u) e^{(-4\lambda^2 - \mu^2 + 3i\lambda + 2)t} \right] = e^{2t}(x + 3t)$$

同样

$$\begin{aligned} F^{-1} \left[G(\lambda, \mu) e^{(-4\lambda^2 - \mu^2)t} \right] &= 5e^{-x^2-y^2} * \left(\frac{1}{2\sqrt{\pi t}} \right) \left(\frac{1}{4\sqrt{\pi t}} \right) \exp\left\{-\frac{x^2}{16t} - \frac{y^2}{4t}\right\} \\ e^{-x^2} * \frac{1}{4\sqrt{\pi t}} \exp\left\{-\frac{x^2}{16t}\right\} &= \frac{1}{\sqrt{1+16t}} e^{-\frac{1}{1+16t}x^2}, \quad e^{-y^2} * \frac{1}{2\sqrt{\pi t}} \exp\left\{-\frac{y^2}{4t}\right\} = \frac{1}{\sqrt{1+4t}} e^{-\frac{1}{1+4t}y^2} \end{aligned}$$

这样

$$F^{-1} \left[G(\lambda, \mu) e^{(-4\lambda^2 - \mu^2)t} \right] = \frac{5}{8\pi t} \frac{1}{\sqrt{1+16t}} \frac{1}{\sqrt{1+4t}} e^{-\frac{1}{1+16t}x^2 - \frac{1}{1+4t}y^2}$$

进一步

$$F^{-1} \left[G(\lambda, \mu) e^{(-4\lambda^2 - \mu^2 + 3i\lambda + 2)t} \right] = \frac{5}{8\pi t} \frac{e^{2t}}{\sqrt{1+16t}} \frac{1}{\sqrt{1+4t}} e^{-\frac{1}{1+16t}(x+3t)^2 - \frac{1}{1+4t}y^2}$$

综上, 求得 $u(t, x, y)$ 为

$$u(t, x, y) = e^{2t}(x+3t) + \frac{5}{8\pi t} \frac{e^{2t}}{\sqrt{1+16t}} \frac{1}{\sqrt{1+4t}} e^{-\frac{1}{1+16t}(x+3t)^2 - \frac{1}{1+4t}y^2}$$

..... 16分

七. 解: 使用球坐标下 Laplace 方程轴对称情形下的求解公式:

$$u = \sum_{n=0}^{+\infty} \left(A_n r^n + B_n r^{-(n+1)} \right) P_n(\cos \theta)$$

..... 4分

利用边界条件得到:

$$u|_{r=1} = \sum_{n=0}^{+\infty} (A_n + B_n) P_n(\cos \theta) = 2 \cos^2 \theta - 2$$

$$u|_{r=2} = \sum_{n=0}^{+\infty} \left(A_n 3^n + B_n 3^{-(n+1)} \right) P_n(\cos \theta) = 2021 + \cos \theta$$

利用变换 $x = \cos \theta$, 并在以上两式比较系数得到:

$$(A_0 + B_0)P_0(x) + (A_1 + B_1)P_1(x) + (A_2 + B_2)P_2(x) = 2x^2 - 2$$

$$(A_0 + \frac{B_0}{3})P_0(x) + (3A_1 + \frac{B_1}{9})P_1(x) + (9A_2 + \frac{B_2}{27})P_2(x) = 21 + x$$

又根据 $P_n(x)$ 的表示, 算得

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1)$$

因此

$$(A_0 + B_0)P_0(x) + (A_1 + B_1)P_1(x) + (A_2 + B_2)P_2(x) = -\frac{4}{3}P_0(x) + \frac{4}{3}P_2(x)$$

$$(A_0 + \frac{B_0}{3})P_0(x) + (3A_1 + \frac{B_1}{9})P_1(x) + (9A_2 + \frac{B_2}{27})P_2(x) = 21P_0(x) + P_1(x)$$

于是得到:

$$A_0 + B_0 = -\frac{4}{3}, A_0 + \frac{B_0}{3} = 21, A_1 + B_1 = 0, 3A_1 + \frac{B_1}{9} = 1, A_2 + B_2 = \frac{4}{3}, 9A_2 + \frac{B_2}{27} = 0$$

解得:

$$A_0 = \frac{193}{6}, B_0 = -\frac{67}{2}, A_1 = \frac{9}{26}, B_1 = -\frac{9}{26}, A_2 = -\frac{2}{363}, B_2 = \frac{162}{121}$$

所以

$$u(r, \theta) = \frac{193}{6} - \frac{67}{2}r^{-1} + \left(\frac{9}{26}r - \frac{9}{26}r^{-2}\right)P_1(\cos \theta) + \left(-\frac{2}{363}r^2 + \frac{162}{121}r^{-3}\right)P_2(\cos \theta)$$

10分

八 解: 作坐标变换 $\bar{x} = x - 3$, $\bar{y} = \frac{y}{2}$, 这样原问题变为

$$\begin{cases} u_{\bar{x}\bar{x}} + 4u_{\bar{y}\bar{y}} = 0, & \bar{x} > 0, \bar{y} > 0, \\ u = \begin{cases} g(2\bar{y}) & \text{当 } \bar{x} = 0, \bar{y} > 0 \\ f(\bar{x} + 3) & \text{当 } \bar{x} > 0, \bar{y} = 0 \end{cases} \end{cases}$$

利用镜像法, 在 $M_0 = (\bar{\xi}, \bar{\eta})$, $M_1 = (\bar{\xi}, -\bar{\eta})$, $M_2 = (-\bar{\xi}, \bar{\eta})$, $M_3 = (-\bar{\xi}, -\bar{\eta})$ 依次放 $+\epsilon, -\epsilon, -\epsilon, +\epsilon$ 的线电荷, 产生电场电势叠加就是区域 $(\bar{x} > 0, \bar{y} > 0)$ 的格林函数:

$$\begin{aligned} \bar{G} = & \frac{1}{2\pi} \ln \sqrt{(\bar{x} + \bar{\xi})^2 + (\bar{y} - \bar{\eta})^2} + \frac{1}{2\pi} \ln \sqrt{(\bar{x} - \bar{\xi})^2 + (\bar{y} + \bar{\eta})^2} \\ & - \frac{1}{2\pi} \ln \sqrt{(\bar{x} - \bar{\xi})^2 + (\bar{y} - \bar{\eta})^2} - \frac{1}{2\pi} \ln \sqrt{(\bar{x} + \bar{\xi})^2 + (\bar{y} + \bar{\eta})^2} \end{aligned}$$

4分

对于区域在 $\bar{y} = 0$ 的边界, 其外法方向 $\vec{n}_0 = (0, -1)$, 则 $\frac{\partial G}{\partial \vec{n}_0} \Big|_{\bar{\eta}=0} = -\frac{\partial G}{\partial \bar{\eta}} \Big|_{\bar{\eta}=0}$,

$$\begin{aligned} \frac{\partial G}{\partial \bar{\eta}} \Big|_{\bar{\eta}=0} &= \frac{1}{4\pi} \left(\frac{(\bar{\eta} - \bar{y})}{(\bar{x} + \bar{\xi})^2 + (\bar{y} - \bar{\eta})^2} + \frac{(\bar{\eta} + \bar{y})}{(\bar{x} - \bar{\xi})^2 + (\bar{y} + \bar{\eta})^2} \right) \Big|_{\bar{\eta}=0} \\ &\quad - \frac{1}{4\pi} \left(\frac{(\bar{\eta} - \bar{y})}{(\bar{x} - \bar{\xi})^2 + (\bar{y} - \bar{\eta})^2} + \frac{(\bar{\eta} + \bar{y})}{(\bar{x} + \bar{\xi})^2 + (\bar{y} + \bar{\eta})^2} \right) \Big|_{\bar{\eta}=0} \\ &= \frac{1}{2\pi} \left(\frac{\bar{y}}{(\bar{x} - \bar{\xi})^2 + \bar{y}^2} - \frac{\bar{y}}{(\bar{x} + \bar{\xi})^2 + \bar{y}^2} \right) \\ \frac{\partial G}{\partial \bar{\xi}} \Big|_{\bar{\xi}=0} &= \frac{1}{2\pi} \left(\frac{\bar{x}}{\bar{x}^2 + (\bar{y} - \bar{\eta})^2} - \frac{\bar{x}}{\bar{x}^2 + (\bar{y} + \bar{\eta})^2} \right) \end{aligned}$$

因此

$$\begin{aligned} u(\bar{x}, \bar{y}) &= \int_{\substack{\bar{\eta}=0 \\ (\bar{\xi}>0)}} f(\bar{\xi} + 3) \frac{\partial G}{\partial \bar{\eta}} d\bar{\xi} + \int_{\substack{\bar{\xi}=0 \\ (\bar{\eta}>0)}} g(2\bar{\eta}) \frac{\partial G}{\partial \bar{\xi}} d\bar{\eta} \\ &= \frac{1}{2\pi} \int_0^{+\infty} f(\bar{\xi} + 3) \left(\frac{\bar{y}}{(\bar{x} - \bar{\xi})^2 + \bar{y}^2} - \frac{\bar{y}}{(\bar{x} + \bar{\xi})^2 + \bar{y}^2} \right) d\bar{\xi} \\ &\quad + \frac{1}{2\pi} \int_0^{+\infty} g(2\bar{\eta}) \left(\frac{\bar{x}}{\bar{x}^2 + (\bar{y} - \bar{\eta})^2} - \frac{\bar{x}}{\bar{x}^2 + (\bar{y} + \bar{\eta})^2} \right) d\bar{\eta} \end{aligned}$$

最后, 利用 $\bar{x} = x - 3$, $\bar{y} = \frac{y}{2}$, 得到:

$$\begin{aligned} u(x, y) &= \frac{1}{\pi} \int_0^{+\infty} f(\xi + 3) \left(\frac{y}{4(x - 3 - \xi)^2 + y^2} - \frac{y}{4(x - 3 + \xi)^2 + y^2} \right) d\xi \\ &\quad + \frac{2}{\pi} \int_0^{+\infty} g(2\eta) \left(\frac{x - 3}{4(x - 3)^2 + (y - 2\eta)^2} - \frac{x - 3}{4(x - 3)^2 + (y + 2\eta)^2} \right) d\eta. \end{aligned}$$

10分