矛盾。

数分 A2 作业参考答案 2022

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1 Week 1

8.1.2 $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, 证明余弦定理

$$||\boldsymbol{x}-\boldsymbol{y}||^2 = ||\boldsymbol{x}||^2 + ||\boldsymbol{y}||^2 - 2||\boldsymbol{x}|| \cdot ||\boldsymbol{y}|| \cos\theta.$$

解.

$$\begin{aligned} ||\boldsymbol{x}-\boldsymbol{y}||^2 &= \langle \boldsymbol{x}-\boldsymbol{y}, \boldsymbol{x}-\boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{x} \rangle - \langle \boldsymbol{y}, \boldsymbol{x} \rangle - \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{y}, \boldsymbol{y} \rangle \\ &= ||\boldsymbol{x}||^2 + ||\boldsymbol{y}||^2 - 2\langle \boldsymbol{x}, \boldsymbol{y} \rangle = ||\boldsymbol{x}||^2 + ||\boldsymbol{y}||^2 - 2||\boldsymbol{x}|| \cdot ||\boldsymbol{y}|| \cos\theta. \end{aligned}$$

□

8.1.4 $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, 证明平行四边形定理

$$||\boldsymbol{x}+\boldsymbol{y}||^2 + ||\boldsymbol{x}-\boldsymbol{y}||^2 = 2\big(||\boldsymbol{x}||^2 + ||\boldsymbol{y}||^2\big).$$

解.

$$\begin{aligned} ||\boldsymbol{x}+y||^2 &= \langle \boldsymbol{x}-\boldsymbol{y}, \boldsymbol{x}-\boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{x} \rangle + \langle \boldsymbol{y}, \boldsymbol{x} \rangle + \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{y}, \boldsymbol{y} \rangle \\ &= ||\boldsymbol{x}||^2 + ||\boldsymbol{y}||^2 + 2\langle \boldsymbol{x}, \boldsymbol{y} \rangle = ||\boldsymbol{x}||^2 + ||\boldsymbol{y}||^2 + 2||\boldsymbol{x}|| \cdot ||\boldsymbol{y}|| \cos\theta. \end{aligned}$$

结合 8.1.2 有

$$||\boldsymbol{x}+\boldsymbol{y}||^2 + ||\boldsymbol{x}-\boldsymbol{y}||^2 = 2\big(||\boldsymbol{x}||^2 + ||\boldsymbol{y}||^2\big).$$

□

8.1.5 $\boldsymbol{a}, \boldsymbol{b}$ 是欧氏空间中两个不同的点，记 $2r = ||\boldsymbol{a}-\boldsymbol{b}|| > 0$, 求证

$$B_r(\boldsymbol{a}) \cap B_r(\boldsymbol{b}) = \emptyset$$

解. 反证，假设 $\exists \boldsymbol{x} \in B_r(\boldsymbol{a}) \cap B_r(\boldsymbol{b})$, 则

$$||\boldsymbol{x}-\boldsymbol{a}|| < r, ||\boldsymbol{x}-\boldsymbol{b}|| < r$$

由向量空间的绝对值不等式

$$2r = ||\boldsymbol{a}-\boldsymbol{b}|| \leq ||\boldsymbol{a}-\boldsymbol{x}|| + ||\boldsymbol{x}-\boldsymbol{b}|| < r + r = 2r,$$

矛盾。 □

8.1.6 $\boldsymbol{x} = (x_1, \cdots, x_n)$, 证明：$\forall \boldsymbol{x} \in \mathbb{R}^n$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} |x_i| \leq ||\boldsymbol{x}||_2 \leq \sum_{i=1}^{n} |x_i|.$$

解. 由 Cauchy 不等式

$$\left(\sum_{i=1}^{n} |x_i| \right)^2 \leq n \sum_{i=1}^{n} |x_i|^2 = n||\boldsymbol{x}||^2$$

得到左边的 $\leq$, 右边的 $\leq$ 平方展开后即得。 □

8.1.7 $\boldsymbol{x} = (x_1, \cdots, x_n)$, 证明：$\forall \boldsymbol{x} \in \mathbb{R}^n$,

$$\max_i |x_i| \leq ||\boldsymbol{x}||_2 \leq \sqrt{n} \max_i |x_i|.$$

解.

$$\max_i |x_i|^2 \leq \sum_{i=1}^{n} |x_i|^2 \leq n \max_i |x_i|^2$$

开方即得。 □

8.2.1 在 $\mathbb{R}^2$ 中定义

$$\boldsymbol{x}_n = \left(\frac{1}{n}, \sqrt[n]{n} \right)$$

, 求证 $\lim\limits_{n \to \infty} \boldsymbol{x}_n = (0, 1)$.

解.

$$\lim_{n \to \infty} \frac{1}{n} = 0,$$
$$\lim_{n \to \infty} \sqrt[n]{n} = 1.$$

由逐分量收敛,

$$\lim_{n \to \infty} \boldsymbol{x}_n = (0, 1)$$

分量的极限通过单变量的极限证明。 □

8.2.2 证明定理 8.2.1.

解. 使用逐分量收敛或定义证明均可。 □

8.2.3* 证明欧氏空间中收敛列有界。

解. 设 $\boldsymbol{x}_n$ 是欧氏空间中的任一收敛列，收敛于点 $\boldsymbol{x}$, 则由定义，$\forall \varepsilon > 0, \exists N > 0$, when $n > N, ||\boldsymbol{x}_n - \boldsymbol{x}|| < \varepsilon$.

对于 $n \leq N, \exists\ 0 < A < \infty, s.t. ||\boldsymbol{x}_n - \boldsymbol{x}|| < A$.

取 $R = \max\{A, \varepsilon\}$ 得 $||\boldsymbol{x}_n - \boldsymbol{x}|| < R, \forall n$, 故 $\{\boldsymbol{x}_n\}$ 有界。 □

8.2.4 证明欧氏空间中基本列有界。

解.（法 1.）欧氏空间中基本列 $\Leftrightarrow$ 收敛列，由 8.2.3 得到基本列收敛。（需要将 8.2.3 结论的证明写出）

（法 2.）设 $\boldsymbol{x}_n$ 是欧氏空间中的任一基本列，则由基本列的定义 $\forall\varepsilon>0,\exists N>0$, when $m,n>N, ||\boldsymbol{x}_m-\boldsymbol{x}_n||<\varepsilon$. 特别地，$||\boldsymbol{x}_{N+1}-\boldsymbol{x}_n||<\varepsilon,\forall n>N$.

另一方面，对于 $n\le N,\exists\, 0<A<\infty, s.t. ||\boldsymbol{x}_n-\boldsymbol{x}_{N+1}||<A$.

取 $R=\max\{A,\varepsilon\}$ 得 $||\boldsymbol{x}_n-\boldsymbol{x}_{N+1}||<R,\forall n$, 故 $\{\boldsymbol{x}_n\}$ 有界。 □

8.3.1 求出 $A^\circ,\bar{A},\partial A$.

space	$\boldsymbol{A}$	$\boldsymbol{A^\circ}$	$\boldsymbol{\bar{A}}$	$\boldsymbol{\partial A}$
$\mathbb{R}$	$\boldsymbol{\{1,\frac{1}{2},\frac{1}{3},\cdots\}}$	$\emptyset$	$A\cup\{0\}$	$A\cup\{0\}$
$\mathbb{R}^2$	$\boldsymbol{\{(x,y):0<y<x+1\}}$	A	$\{(x,y):0\le y\le x+1\}$	$\{(x,y):x\ge -1$且$y=0$ or $y=x+1\}$
$\mathbb{R}^n$	**有限点集**	$\emptyset$	A	A

8.3.2 设 $A=\{(x,y):x,y\in\mathbb{Q}\}$, 求 $A^\circ,(A^c)^\circ,\partial A$.

解. 由有理数的稠密性知

(1)$A^\circ=\emptyset$. 对每一个 A 中的点 $\boldsymbol{x}$, 它的每一个邻域内都有分量有无理数的点，即 $\forall\boldsymbol{x}\in A,\forall r,B_r(\boldsymbol{x})\cup A^c\neq\emptyset$.

(2)$(A^c)^\circ=\emptyset$. 同 (1).

(3)$\partial A=\mathbb{R}^2.\mathbb{R}^2$ 中的每一个点的任意邻域内都有 A 中的点, 或 $\partial A=\mathbb{R}^2\backslash(A^\circ\cup(A^c)^\circ)=\mathbb{R}^2$. □

8.3.3* 证明 $\boldsymbol{x}\in\bar{A}\Leftrightarrow\forall r>0,B_r(\boldsymbol{x})\cap A\neq\emptyset$.

解. "$\Rightarrow$"：

$$\begin{aligned}&\boldsymbol{x}\in\bar{A}=A\cup A',\\&\text{if }\boldsymbol{x}\in A\Rightarrow\boldsymbol{x}\in B_r(\boldsymbol{x})\cap A\neq\emptyset,\forall r>0.\\&\text{if }\boldsymbol{x}\in A'\overset{\text{def}}{\Rightarrow}\forall r>0,B_r(\check{\boldsymbol{x}})\cap A\neq\emptyset\Rightarrow B_r(\boldsymbol{x})\cap A\neq\emptyset,\forall r>0.\end{aligned}$$

"$\Leftarrow$"：

$$\begin{aligned}&\forall r>0,B_r(\boldsymbol{x})\cap A\neq\emptyset,\\&\text{if }\boldsymbol{x}\in A\Rightarrow\boldsymbol{x}\in A\subset\bar{A}.\\&\text{if }\boldsymbol{x}\notin A\Rightarrow\forall r>0,B_r(\check{\boldsymbol{x}})\cap A\neq\emptyset\overset{\text{def}}{\Rightarrow}\boldsymbol{x}\text{是}A\text{的聚点}\Rightarrow\boldsymbol{x}\in A'\subset\bar{A}.\end{aligned}$$

□

8.3.5 证明 $\partial A=\bar{A}\cap(A^\circ)^c$.

解.

$$\boldsymbol{x}\in\partial A\Leftrightarrow\forall r>0,\begin{cases}B_r(\boldsymbol{x})\cap A\neq\emptyset\overset{8.3.3}{\Leftrightarrow}\boldsymbol{x}\in\bar{A}\\B_r(\boldsymbol{x})\cap A^c\neq\emptyset\Leftrightarrow B_r(\boldsymbol{x})\not\subseteq A\Leftrightarrow\boldsymbol{x}\notin A^\circ\end{cases}\Leftrightarrow\boldsymbol{x}\in\bar{A}\cap(A^\circ)^c.$$

□

8.3.7 证明 (1)$(A\cap B)^\circ=A^\circ\cap B^\circ$.(2)$\overline{A\cup B}=\bar{A}\cup\bar{B}$.

解. (1)

$$\boldsymbol{x}\in(A\cap B)^\circ\Leftrightarrow\exists r>0,B_r(\boldsymbol{x})\subset A\cap B\Leftrightarrow\begin{cases}B_r(\boldsymbol{x})\subset A\Leftrightarrow x\in A^\circ\\ B_r(\boldsymbol{x})\subset B\Leftrightarrow x\in B^\circ\end{cases}\Leftrightarrow\boldsymbol{x}\in A^\circ\cap B^\circ.$$

(2)

(a) $\overline{A\cup B}\subset\bar{A}\cup\bar{B}$

$$\begin{aligned}\boldsymbol{x}\in\overline{A\cup B}\Rightarrow\ &\boldsymbol{x}\in A\cup B\subset\bar{A}\cup\bar{B}\\ &\text{or } \boldsymbol{x}\in(A\cap B)'\Rightarrow\exists\{\boldsymbol{x}_n\}_{n=1}^\infty\subset A\cup B, s.t.\boldsymbol{x}_n\to\boldsymbol{x}\\ &\Rightarrow\begin{cases}\{\boldsymbol{x}_n\}\text{中只有有限项不同}\Rightarrow\boldsymbol{x}\in A\cup B\subset\bar{A}\cup\bar{B}\\ \{\boldsymbol{x}_n\}\text{中有无限项在 } A \text{ 中或者 } B \text{ 中}\Rightarrow\boldsymbol{x}\in A'\subset\bar{A}\text{ or }\boldsymbol{x}\in B'\subset\bar{B}\end{cases}\\ &\Rightarrow\boldsymbol{x}\in\bar{A}\cup\bar{B}.\end{aligned}$$

(b) $\bar{A}\cup\bar{B}\subset\overline{A\cup B}$

$$\boldsymbol{x}\in\bar{A}\cup\bar{B}\Rightarrow\boldsymbol{x}\in\bar{A}\subset\overline{A\cup B}\text{ or }\boldsymbol{x}\in\bar{B}\subset\overline{A\cup B}$$

□

2 Week 2

8.3.8 (1) 作出闭集列 $\{F_i\}$, 使得 $\bigcup\limits_{i=1}^\infty F_i=B_1(\mathbf{0})$;(2) 作出开集列 $\{G_i\}$, 使得 $\bigcap\limits_{i=1}^\infty G_i=B_1(\mathbf{0})$.

解.

$$F_i=\{\boldsymbol{x}\in\mathbb{R}^n:||x||\le 1-\frac{1}{i}\}=\overline{B_{1-\frac{1}{i}}(\mathbf{0})};$$
$$G_i=\{\boldsymbol{x}\in\mathbb{R}^n:||x||<1+\frac{1}{i}\}=B_{1+\frac{1}{i}}(\mathbf{0}).$$

□

8.3.9 I 为指标集，证明：(1)$\overline{\bigcap\limits_{\alpha\in I}A_\alpha}\subset\bigcap\limits_{\alpha\in I}\overline{A_\alpha}$;(2)$\big(\bigcup\limits_{\alpha\in I}A_\alpha\big)^\circ\supset\bigcup\limits_{\alpha\in I}A_\alpha^\circ$. 并举例说明真包含关系是可以出现的。

解. (1)

$$A_\alpha\subset\overline{A_\alpha}\Rightarrow\bigcap_{\alpha\in I}A_\alpha\subset\bigcap_{\alpha\in I}\overline{A_\alpha}\overset{\text{闭包}}{\Rightarrow}\overline{\bigcap_{\alpha\in I}A_\alpha}\subset\overline{\bigcap_{\alpha\in I}\overline{A_\alpha}}\overset{\text{闭集}}{=}\bigcap_{\alpha\in I}\overline{A_\alpha}.$$

举例：$A_1=(0,1),A_2=(1,2).\overline{A_1\cap A_2}=\emptyset,\overline{A_1}\cup\overline{A_2}=\{1\}$.

(2)

$$A_\alpha^\circ\subset A_\alpha\Rightarrow\bigcup_{\alpha\in I}A_\alpha^\circ\subset\bigcap_{\alpha\in I}A_\alpha\overset{\text{内部}}{\Rightarrow}\big(\bigcap_{\alpha\in I}A_\alpha\big)^\circ\supset\big(\bigcap_{\alpha\in I}(A_\alpha)^\circ\big)^\circ\overset{\text{开集}}{=}\bigcap_{\alpha\in I}(A_\alpha)^\circ.$$

举例：$A_1=[0,1],A_2=[1,2].A_1^\circ\cup A_2^\circ=(0,2)\backslash\{1\},(A_1\cup A_2)^\circ=(0,2)$. □

8.3.10 设 $E\in\mathbb{R}^n$. 求证：∂E 是闭集。

解. 根据书上定义，$\partial = \mathbb{R}^n \backslash \big(E^\circ \cup (E^c)^\circ\big)$,$E^\circ \cup (E^c)^\circ$ 是开集，故 ∂E 是闭集。

也可根据集合中点的描述证明。

$$\boldsymbol{x} \in \partial E \Leftrightarrow B_r(\boldsymbol{x}) \cap E \neq \emptyset 且 B_r(\boldsymbol{x}) \cap E^c \neq \emptyset, \forall r > 0.$$

故

$$\boldsymbol{x} \notin \partial E \Leftrightarrow \exists r > 0, B_r(\boldsymbol{x}) \cap E \neq \emptyset \text{ or } B_r(\boldsymbol{x}) \cap E^c \neq \emptyset.$$

所以

$$\forall \boldsymbol{y} \in B_r(\boldsymbol{x}), 取 r' = r - ||\boldsymbol{x} - \boldsymbol{y}||, 则 B_{r'}(\boldsymbol{y}) \subset B_r(\boldsymbol{x}).$$

因此

$$B_{r'}(\boldsymbol{y}) \cap E = \emptyset \text{ or } B_{r'}(\boldsymbol{y}) \cap E^c = \emptyset \Rightarrow y \in (\partial E)^c.$$

$B_r(\boldsymbol{x}) \subset (\partial E)^c, (\partial E)^c$ 是开集。 □

8.3.11 设 $G_1, G_2 \subset \mathbb{R}^n$ 是两个不相交的开集，证明 $G_1 \cap \overline{G_2} = \overline{G_1} \cap G_2 = \emptyset$.

解. 根据 G_1, G_2 的对称性，只需证其中一个。假设 $\exists \boldsymbol{x} \in G_1 \cap \overline{G_2} \neq \emptyset$. 则

$$\boldsymbol{x} \in G_1 \overset{开集}{\Rightarrow} \exists r_0 > 0, B_{r_0}(\boldsymbol{x}) \subset G_1$$
$$\boldsymbol{x} \in \overline{G_2} \Rightarrow \forall r > 0, B_r(\boldsymbol{x}) \cap G_2 \neq \emptyset$$

故存在 $\boldsymbol{y} \in B_{r_0}(\boldsymbol{x}) \cap G_2 \subset G_1 \cap G_2 \neq \emptyset$, 矛盾。 □

8.3.12 P 是投影算子，E 为 $\mathbb{R}^2$ 中的开集，求证 $P(E)$ 是 $\mathbb{R}$ 中的开集。并举例说明若 A 是 $\mathbb{R}^2$ 中的闭集，$P(A)$ 不一定是 $\mathbb{R}$ 中的闭集。

解.

$$\forall x \in P(E), \exists (x, y) \in E, \exists r_x > 0, s.t. B_{r_x}\big((x, y)\big) \subset E$$

则

$$(x - r, x + r) = P\Big(B_{r_x}\big((x, y)\big)\Big) \subset P(E),$$

$P(E)$ 是开集。

举例：$A = \Big\{\big(x, \frac{1}{x}\big) : x > 0\Big\}$ 为闭集，其像 $P(A) = (0, +\infty)$ 非闭集。 □

8.3.13 E 闭集 $\Leftrightarrow \partial E \subset E$.

解. "$\Rightarrow$"：

$$\forall \boldsymbol{x} \in \partial E \Leftrightarrow \forall r > 0, B_r(\boldsymbol{x}) \cap E \neq \emptyset 且 B_r(\boldsymbol{x}) \cap E^c \neq \emptyset.$$

E 是闭集，则 E^c 是开集，故由 $\forall r > 0, B_r(\boldsymbol{x}) \cap E^c \neq \emptyset$ 得到 $x \notin E^c$.

"$\Leftarrow$"：

$$\forall \boldsymbol{x} \in E^c, x \notin \partial E \Leftrightarrow \exists r > 0, B_r(\boldsymbol{x}) \cap E \neq \emptyset \text{ or } B_r(\boldsymbol{x}) \cap E^c \neq \emptyset.$$

由于 $\boldsymbol{x} \in E^c, B_r(x) \cap E^c \neq \emptyset$, 故 $B_r(\boldsymbol{x}) \cap E = \emptyset$, 即 $B_r(\boldsymbol{x}) \subset E^c$, 所以 E^c 是开集，E 是闭集。

□

8.4.1 P 是投影算子，证明若 $A \subset \mathbb{R}^2$ 是紧致集，则 $P(A)$ 是紧致集。

解.（紧致 ⇔ 列紧） 任取 $\{x_n\}_{n=1}^{\infty}\subset P(A)$, $\exists\{(x_n,y_n)\}_{n=1}^{\infty}\subset A$. 由于 A 列紧，故有收敛子列 $\{(x_{n_i},y_{n_i})\}_{i=1}^{\infty}$ 在 A 中收敛到 (x,y). 由于向量的收敛等价于按分量收敛，故 $\{x_{n_i}\}$ 也是 $\{x_n\}$ 的收敛子列，且 $\lim\limits_{i\Rightarrow\infty}x_{n_i}=x=P\big((x,y)\big)\in P(A)$. 故 $P(A)$ 列紧。

（紧致的定义） 要证明 $P(A)$ 的任意开覆盖存在有限子覆盖，设 $\{G_\alpha\}_{\alpha\in I}$ 是 $P(A)$ 的任意一个开覆盖.记 $\widetilde{G_\alpha}=\{(x,y):x\in G_\alpha,y\in\mathbb{R}\}$, 则 $\widetilde{G_\alpha}$ 是开集.$\Big(\forall(x,y)\in\widetilde{G_\alpha}$, 由于 G_α 是开集，故 $\exists r_0,s.t.(x-r_0,x+r_0)\subset G_\alpha, B_{r_0}\big((x,y)\big)\subset(x-r_0,x+r_0)\times(y-r_0,y+r_0)\subset G_\alpha\times\mathbb{R}=\widetilde{G_\alpha}.\Big)$

$\{\widetilde{G_\alpha}\}_{\alpha\in I}$ 是 A 的开覆盖，故存在其有限子覆盖 $\{\widetilde{G_i}\}_{i=1}^{m}$.

$\forall x\in P(A),\exists(x,y)\in A,\exists 1\le i\le m,s.t.(x,y)\in\widetilde{G_i}$, 故 $x\in G_i$.$\{\widetilde{G_i}\}_{i=1}^{m}$ 是 $P(A)$ 的覆盖 $\{G_\alpha\}_{\alpha\in I}$ 的有限子覆盖。故 $P(A)$ 紧致。 □

8.4.2 $A,B\subset\mathbb{R}$. 证明 $A\times B$ 紧致 $\Leftrightarrow A,B$ 紧致。

解. "⇒"：由 8.4.1 知。

"⇐"：利用列紧证明即可。 □

8.4.3 证明 $A\subset\mathbb{R}^n$ 紧致的定义与下述命题等价：若 $\mathfrak{F}=\{A_\alpha\}$ 是 $\mathbb{R}^n$ 中的闭集族且 $A\cap(\bigcap\limits_{\alpha}A_\alpha)=\emptyset$, 则 $\exists A_1,\cdots,A_k\in\mathfrak{F}$, 使 $A\cap(\bigcap\limits_{i=1}^{k}A_i)=\emptyset$.

解. A 紧致等价于任意 A 的开覆盖 $\{G_\alpha\}$ 存在有限子覆盖 $G_1,\cdots,G_k,A\subset\bigcup\limits_{i=1}^{k}G_i$.

"⇒"：任意满足 $A\cap(\bigcap\limits_{\alpha}A_\alpha)=\emptyset$ 的闭集族 $\{A_\alpha\}$, 有 $A\subset\big(\bigcap\limits_{\alpha}A_\alpha\big)^c=\bigcup\limits_{\alpha}A_\alpha^c$. 故 $\{A_\alpha^c\}$ 是 A 的开覆盖。由 A 紧致，故存在有限子覆盖，$\exists A_1,\cdots,A_k,A\subset\bigcup\limits_{i=1}^{k}A_i^c=\big(\bigcap\limits_{i=1}^{k}A_i\big)^c$, 即 $A\cap(\bigcap\limits_{i=1}^{k}A_i)=\emptyset$.

"⇐"：设 $\{G_\alpha\}$ 是 A 的任一开覆盖，$A\subset\bigcup\limits_{\alpha}G_\alpha$, 则 $\emptyset=A\cap\big(\bigcup\limits_{\alpha}G_\alpha\big)^c=A\cap\big(\bigcap\limits_{\alpha}G_\alpha^c\big)$. 则 $\{G_\alpha^c\}$ 是满足明题条件的闭集族，故 $\exists G_1,\cdots,G_k,s.t.A\cap\big(\bigcap\limits_{i=1}^{k}G_i^c\big)=\emptyset$, 即 $A\subset\big(\bigcap\limits_{i=1}^{k}G_i^c\big)^c=\bigcup\limits_{i=1}^{k}G_i$, 故 $\{G_\alpha\}$ 存在有限子覆盖，A 紧致。 □

8.4.4 证明 Frechet 紧等价于列紧。$A\subset\mathbb{R}^n$ 称为 Frechet 紧，若 A 的每一个无穷子集在 A 中有一个凝聚点。

解. "⇒"：设 $\{\boldsymbol{x}_n\}_{n=1}^{\infty}\subset A$ 为任意点列。则 $\{\boldsymbol{x}_n\}$ 作为 A 的子集，若 $\{\boldsymbol{x}_n\}$ 为有限集合 (即只有有限个不同项)，显然有收敛子列。故只需考虑 $\{\boldsymbol{x}_n\}$ 为无穷子集的情况，由于 AFrechet 紧，$\{\boldsymbol{x}_n\}$ 在 A 中有凝聚点 $\boldsymbol{x}$, 即存在子列收敛到 $\boldsymbol{x}$. 故 A 列紧。

"⇐"：任取 A 的无穷子集 E, 将其看作 A 中点列。则由 A 列紧，E 中必有收敛子列 $\{\boldsymbol{x}_n\}\Rightarrow\boldsymbol{x}\in A$, 且 $\boldsymbol{x}_n\neq\boldsymbol{x},\forall n$, 故 $\forall r>0,\exists\boldsymbol{x}_N,s.t.\boldsymbol{x}_N\in E\cap B_r(\check{\boldsymbol{x}})\neq\emptyset$, 故 $\boldsymbol{x}$ 是 E 的凝聚点。A 是 Frechet 紧。 □

8.4.5 设 $F_1,\cdots,F_k,\cdots$ 是 $\mathbb{R}^n$ 中的非空闭集族/非空紧致集族，满足 $F_k\supset F_{k+1},k\ge1$, 是否一定有 $\bigcap\limits_{k=1}^{\infty}F_k\neq\emptyset$.

解. 若是闭集，结论不一定成立，反例：在 $\mathbb{R}$ 中，$F_k=[k,+\infty)$, $\bigcap\limits_{k=1}^{\infty}F_k=\emptyset$.

若是紧致集，$\forall k\ge1,\exists\boldsymbol{x}_k\in F_k$, 则 $\{\boldsymbol{x}_k\}_{k=1}^{\infty}\subset F_1$ 是有界列，由 Bolzano-Weierstrass 定理，存在子列 $\{\boldsymbol{x}_{k_i}\}$ 趋于 $\boldsymbol{x}$, 且 $\boldsymbol{x}_{k_i}\in\bigcap\limits_{l=1}^{k_i}F_l$, 使 $i\to\infty,\boldsymbol{x}\in\bigcap\limits_{l=1}^{\infty}F_l\neq\emptyset$. □

8.5.2 $A \subset \mathbb{R}^n$. 若 A 既开又闭，求证 $A = \mathbb{R}^n$ 或 $A = \emptyset$.

解. 容易证明 $\mathbb{R}^n$ 和 $\emptyset$ 均满足条件。用反证法证明除此之外没有别的集合满足既开又闭。假设 A 既开又闭，且 $A \neq \mathbb{R}^n, A \neq \emptyset$. 则 $\mathbb{R}^n = A \cup A^c$ 是 $\mathbb{R}^n$ 的剖分，且满足 $A \neq \emptyset, A^c \neq \emptyset, A$ 和 A^c 均为开集，而 $\mathbb{R}^n$ 是连通集，矛盾。 □

8.6.3 计算极限。

(1) $\lim\limits_{(x,y)\to(0,0)} \left(x^2+y^2\right)^{x^2y^2}$;

解.

$$\left|x^2y^2\log\left(x^2+y^2\right)\right| \le \left(x^2+y^2\right)^2\left|\log\left(x^2+y^2\right)\right| \to 0, \text{ as } (x,y)\to(0,0).$$

故 $\lim\limits_{(x,y)\to(0,0)} \left(x^2+y^2\right)^{x^2y^2} = \lim\limits_{(x,y)\to(0,0)} \exp\left(\left|x^2y^2\log\left(x^2+y^2\right)\right|\right) = 1.$ □

(2) $\lim\limits_{\substack{x\to+\infty\\y\to+\infty}} \left(\frac{xy}{x^2+y^2}\right)^{x^2}$;

解.

$$0 < \frac{xy}{x^2+y^2} \le \frac{(x^2+y^2)/2}{x^2+y^2} = \frac{1}{2} < 1,$$

$$0 < \left(\frac{xy}{x^2+y^2}\right)^{x^2} \le \left(\frac{1}{2}\right)^{x^2} \to 0 \text{ as } x\to+\infty,$$

故 $\lim\limits_{\substack{x\to+\infty\\y\to+\infty}} \left(\frac{xy}{x^2+y^2}\right)^{x^2} = 0,$ □

(3) $\lim\limits_{\substack{x\to+\infty\\y\to+\infty}} \left(x^2+y^2\right)e^{-(x+y)}$.

解.

$$0 < \left(x^2+y^2\right)e^{-(x+y)} < (x+y)^2e^{-(x+y)} \to 0,$$

故 $\lim\limits_{\substack{x\to+\infty\\y\to+\infty}} \left(x^2+y^2\right)e^{-(x+y)} = 0.$ □

8.6.5 (1) 求 $f(x,y) = \frac{x^2y}{x^4+y^2}$ 在 $(0,0)$ 处的两个累次极限。

解.

$$\lim_{x\to0}\lim_{y\to0}\frac{x^2y}{x^4+y^2} = 0, \lim_{y\to0}\lim_{x\to0}\frac{x^2y}{x^4+y^2} = 0.$$

□

(2) 计算 $\lim\limits_{x\to\infty}\lim\limits_{y\to\infty}\sin\left(\frac{\pi x}{2x+y}\right), \lim\limits_{y\to\infty}\lim\limits_{x\to\infty}\sin\left(\frac{\pi x}{2x+y}\right)$.

解.

$$\lim_{x\to\infty}\lim_{y\to\infty}\sin\left(\frac{\pi x}{2x+y}\right) = 0, \lim_{y\to\infty}\lim_{x\to\infty}\sin\left(\frac{\pi x}{2x+y}\right) = 1.$$

□

(3) 计算 $\lim\limits_{x\to+\infty}\lim\limits_{y\to0^+}\frac{x^y}{1+x^y}, \lim\limits_{y\to0^+}\lim\limits_{x\to+\infty}\frac{x^y}{1+x^y}$.

解.

$$\lim_{x\to+\infty}\lim_{y\to0^+}\frac{x^y}{1+x^y}=\frac{1}{2},\lim_{y\to0^+}\lim_{x\to+\infty}\frac{x^y}{1+x^y}=1.$$

□

8.6.6 $f(x,y)=(x+y)\sin(1/x)\sin(1/y)$. 证明 2 个累次极限均不存在, 但 $\lim_{(x,y)\to(0,0)}f(x,y)=0$.

解.

$$\lim_{y\to0}(x+y)\sin\frac{1}{x}\sin\frac{1}{y}=\lim_{y\to0}x\sin\frac{1}{x}\sin\frac{1}{y}$$

极限不存在，另一个累次极限同理，故 2 个累次极限均不存在。

$$0<\left|(x+y)\sin\frac{1}{x}\sin\frac{1}{y}\right|\le|x+y|\to0 \text{ as } (x,y)\to(0,0),$$

得 $\lim_{(x,y)\to(0,0)}f(x,y)=0$. □

8.6.7 设

$$\lim_{\substack{x\to x_0\\y\to y_0}}f(x,y)=a$$

存在，又对 y_0 近旁得每一个 y, 极限 $\lim_{x\to x_0}f(x,y)=h(y)$ 存在，证明 $\lim_{y\to y_0}h(y)=a$.

解. $\lim_{\substack{x\to x_0\\y\to y_0}}f(x,y)=a$ 存在，故 $\forall\varepsilon>0,\exists\delta>0,||(x,y)-(x_0,y_0)||<\delta$ 时，有 $|h(y)-a|<\varepsilon$. 对不等式两边取上极限,

$$|h(y)-a|=\overline{\lim_{x\to x_0}}|f(x,y)-a|\le\varepsilon$$

对 $||y-y_0||<\delta$ 成立，得 $\lim_{y\to y_0}h(y)=a$. □

8.6.8 $f(x,y)$ 在某点处得 2 个累次极限和极限都存在，则这 3 个值相等。

解. 由累次极限定义和 8.6.7 即可得到结论。 □

3 Week 3

8.7.2 设

$$f(x,y)=\frac{1}{1-xy},(x,y)\in[0,1]^2\backslash\{(1,1)\},$$

求证:f 连续但不一致连续

解. 由 $1-xy$ 连续且其不为 0，则由极限的四则运算及得 $f(x,y)$ 连续。取 $(x_n,y_n)=(1-\frac{1}{n},1-\frac{1}{n}),(x_n',y_n')=(1-\frac{1}{n},1)$,

$$|f(x_n,y_n)-f(x_n',y_n')|=\left|\frac{n-n^2}{2n-1}\right|\to+\infty \text{ as } n\to+\infty$$

故其不一致连续。 □

8.7.3 设 $A \subset \mathbb{R}^n$, $\boldsymbol{p} \in \mathbb{R}^n$. 定义

$$\rho(\boldsymbol{p}, A) = \inf_{\boldsymbol{a} \in A} ||\boldsymbol{p} - \boldsymbol{a}||,$$

称之为点 $\boldsymbol{p}$ 到集合 A 的距离，证明：

(1) 若 $A \neq \varnothing$, 则 $\bar{A} = \{\boldsymbol{p} \in \mathbb{R}^n : \rho(\boldsymbol{p}, A) = 0\}$;

(2) 对任意的 $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^n$, 有

$$|\rho(\boldsymbol{p}, A) - \rho(\boldsymbol{q}, A)| \leq ||\boldsymbol{p} - \boldsymbol{q}||.$$

这说明 $\rho(\boldsymbol{p}, A)$ 是 $\mathbb{R}^n$ 上的连续函数.

解. (1) “$\subset$” :$\forall \boldsymbol{p} \in \bar{A}$,$\exists \{x_n\} \subset A$ 且 $x_n \to \boldsymbol{p}$, 则 $||x_n - \boldsymbol{p}|| \to 0$, 故 $\rho(\boldsymbol{p}, A) = \inf_{\boldsymbol{a} \in A} ||\boldsymbol{p} - \boldsymbol{a}|| = 0$.
“$\supset$” : 若 $\boldsymbol{p} \notin \bar{A}$, 因为 $\bar{A}$ 为闭集, 则 $\exists r > 0$, 使得 $B_r(X) \cap \bar{A} = \varnothing$, 则 $||\boldsymbol{p} - \boldsymbol{a}|| \geq r, \forall \boldsymbol{a} \in A$, 则

$$\inf_{\boldsymbol{a} \in A} ||\boldsymbol{p} - \boldsymbol{x}|| \geq r, i.e. \rho(\boldsymbol{p}, A) \neq 0.$$

(2) 不妨设 $\rho(\boldsymbol{p}, A) \leq \rho(\boldsymbol{q}, A)$ 则只用证明 $\rho(\boldsymbol{p}, A) \leq \rho(\boldsymbol{q}, A) + ||\boldsymbol{p} - \boldsymbol{q}||$. 由 $\rho(\boldsymbol{p}, A) \leq ||\boldsymbol{p} - \boldsymbol{a}|| \leq ||\boldsymbol{a} - \boldsymbol{q}|| + ||\boldsymbol{q} - \boldsymbol{p}|| \quad \forall \boldsymbol{a} \in A$. 对右边取 inf，我们有 $\rho(\boldsymbol{p}, A) \leq \rho(\boldsymbol{p}, A) + ||\boldsymbol{p} - \boldsymbol{q}||$. 故得证。 □

8.7.4 设 $A, B \subset \mathbb{R}^n$. 定义

$$\rho(A, B) = \inf\{||\boldsymbol{p} - \boldsymbol{q}|| : \boldsymbol{p} \in A, \boldsymbol{q} \in B\}$$

称之为集合 A 和 B 之间的距离, 证明：

(1) 若 A 为紧致集，则存在一点 $\boldsymbol{a} \in A$, 使得 $\rho(\boldsymbol{a}, B) = \rho(A, B)$;

(2) 若 A, B 为紧致集，则存在一点 $\boldsymbol{a} \in A$,$\boldsymbol{b} \in B$, 使得 $||\boldsymbol{a} - \boldsymbol{b}|| = \rho(A, B)$;

(3) 设 A 为紧致集，B 为闭集，则 $\rho(A, B) = 0$ 当且仅当 $A \cap B \neq \varnothing$.

解. (1) 由 $\rho(\boldsymbol{a}) = \rho(\boldsymbol{a}, B)$ 连续，且 A 为紧致集，则 ρ 可以在 A 上面达到最小值 $\boldsymbol{a}$. 且由范数的连续性，二重极限和累次极限相等。故我们有

$$\rho(\boldsymbol{a}, B) = \min_{\boldsymbol{p} \in A} \rho(\boldsymbol{p}, B) = \inf_{\boldsymbol{p} \in A} \inf_{\boldsymbol{q} \in B} ||\boldsymbol{p} - \boldsymbol{q}|| = \inf_{\boldsymbol{p} \in A, \boldsymbol{q} \in B} ||\boldsymbol{p} - \boldsymbol{q}|| = \rho(A, B).$$

(2) 由 (1)，则存在 $\boldsymbol{a} \in A$, 使得 $\rho(A, B) = \rho(\boldsymbol{a}, B) = \inf_{\boldsymbol{b} \in B} ||\boldsymbol{a} - \boldsymbol{b}||$. 固定 $\boldsymbol{a}$, 则 $\rho(\boldsymbol{b}) = \rho(\boldsymbol{a}, \boldsymbol{b})$ 关于 $\boldsymbol{b}$ 连续，B 为紧致集，故达到最小值 $\boldsymbol{b}$. 故我们有

$$\rho(\boldsymbol{a}, \boldsymbol{b}) = \min_{\boldsymbol{b} \in B} ||\boldsymbol{a} - \boldsymbol{b}|| = \rho(\boldsymbol{a}, B) = \rho(A, B)$$

(3) “$\Leftarrow$”：显然.
“$\Rightarrow$” 由 (1)，$\exists \boldsymbol{a} \in A$ 使得 $\rho(a, B) = \rho(A, B) = 0$, 则 $\exists \boldsymbol{b}_n \in B$ 使得 $||\boldsymbol{a} - \boldsymbol{b}_n|| \to 0$。于是我们有 $\boldsymbol{b}_n \to \boldsymbol{a} \Rightarrow \boldsymbol{a} \in B' \subset B \Rightarrow \boldsymbol{a} \in A \cap B \Rightarrow A \cap B \neq \varnothing$. □

8.7.5 作出两个不相交的闭集 A,B, 使得 $\rho(A, B) = 0$.

解.

$$A = \{(x, y) | xy = 1, x \neq 0\} \qquad B = \{(x, 0) | x \in \mathbb{R}\}.$$

□

8.7.6 设 $A \subset \mathbb{R}^n$. 证明：对任意的常数 $c>0$,$\{\boldsymbol{p}\in\mathbb{R}^n:\rho(\boldsymbol{p},A)\le c\}$ 是紧致集.

解. 由 $\rho(\boldsymbol{p})=\rho(\boldsymbol{p},A)$ 为关于 $\boldsymbol{p}$ 的连续函数. 则 $\{\boldsymbol{p}\in\mathbb{R}^n:\rho(\boldsymbol{p},A)\le c\}=\rho^{-1}(-\infty,c]$ 为闭集. 由 $A\subset\mathbb{R}^n$ 有界. $\Rightarrow \exists r>0$, 使得 $A\subset B_r(0)$. 有三角不等式得

$$||\boldsymbol{p}||\le||\boldsymbol{a}-\boldsymbol{p}||+||\boldsymbol{a}||\le||\boldsymbol{a}-\boldsymbol{p}||+r \quad \forall \boldsymbol{a}\in A$$

对 $\boldsymbol{a}$ 取 inf $\Rightarrow ||\boldsymbol{p}||\le r+C \Rightarrow \{\boldsymbol{p}\in\mathbb{R}^n:\rho(\boldsymbol{p},A)\le c\}\subset B_{r+c}(0)$. 故 $\{\boldsymbol{p}\in\mathbb{R}^n:\rho(\boldsymbol{p},A)\le c\}$ 有界. 由有界闭集则为紧致集. □

8.7.7 设连续函数 $f:\mathbb{R}^n\to\mathbb{R}$ 既取正值，也取负值。求证：集合 $E=\{\boldsymbol{p}\in\mathbb{R}^n:f(\boldsymbol{p})\neq 0\}$ 是非连通集.

解.

$$E=f^{-1}(-\infty,0)\cup f^{-1}(0,+\infty)$$

由 f 连续，则 $f^{-1}(-\infty,0)$, $f^{-1}(0,+\infty)$ 为不相交的非空开集，故 E 不连通. □

8.8.1 设 $f:\mathbb{R}^n\to\mathbb{R}^m$ 连续,$E\subset\mathbb{R}^n$. 求证：$f(\bar{E})\subset\overline{f(E)}$. 在什么条件下有 $f(\bar{E})=\overline{f(E)}$

解. 由 $\overline{f(E)}$ 为闭集，且 f 为连续函数, 则 $f^{-1}(\overline{f(E)})$ 为闭集. 而由定义 $E\subset f^{-1}(\overline{f(E)})$, 两边取闭包有 $\overline{E}\subset f^{-1}(\overline{f(E)})$. 则有 $f(\bar{E})\subset\overline{f(E)}$.
若 E 为紧致集, 则 E 为闭集，且由 f 连续则 $f(E)$ 仍为紧致集则也为闭集. 则有 $E=\overline{E}$, $f(E)=\overline{f(E)}$. 故有 $f(\bar{E})=\overline{f(E)}$. □

8.7.2 设 $E\subset\mathbb{R}, f:E\to\mathbb{R}^n$. 证明:

(1) 若 E 是闭集,f 连续, 则 f 的图像

$$G(f)=\{(x,f(x)):x\in E\}$$

是 $\mathbb{R}^{m+1}$ 中的闭集;

(2) 若 E 是紧致集,f 连续, 则 $G(f)$ 也是紧致集;

(3) 若 $G(f)$ 是紧致集, 则 f 连续.

解. (1)$\forall$ $(x_n,f(x_n))\to(x,y)$, 则 $x_n\to x$ 且 $x_n\in E$, E 为闭集, 则 $x\in E$. 由 f 连续, 则 $f(x_n)\to f(x)$. 而 $f(x_n)\to y\Rightarrow f(x)=y$, 则 $(x,y)=(x,f(x))\in G(f)$.

(2)$g:E\to G(f) \quad g(x)=(x,f(x))$ 则由 f 连续 $\Rightarrow g$ 连续. 由 E 为紧致集，则 $G(f)$ 也为紧致集.

(3) 若 f 在 x_0 点处不连续, 则 $\exists\epsilon_0$, $\exists x_n$ 使得 $|f(x_0)-f(x_n)|>\epsilon_0 \quad n=1,2,...$ 由 $G(f)$ 为紧致集, 则对 $\{(x_n,f(x_n))\}$ 存在收敛子列 $\{(x_{n_k},f(x_{n_k}))\}\to(x,f(x))$ 由 $x_n\to x_0$, 则 $x_0=x\Rightarrow f(x_{n_k})\to f(x_0)$ 矛盾。 □

9.1.2 设函数 $f(x,y)=\sqrt{|x^2-y^2|}$. 在坐标原点处沿着哪些方向 f 的方向导数存在?

解. 令 $v=(\cos\theta,\sin\theta)$, 则

$$\frac{\partial f}{\partial v}(0,0)=\lim_{t\to 0}\frac{\sqrt{(t\cos\theta)^2-(t\sin\theta)^2}}{t}=\lim_{t\to 0}\frac{|t|}{t}\sqrt{|\cos 2\theta|},$$

则只有 $\cos 2\theta = 0$ 才有极限存在。解得

$$\theta = \frac{\pi}{4}\text{或}\frac{3\pi}{4}\text{或}\frac{5\pi}{4}\text{或}\frac{7\pi}{4}$$

□

9.1.3 设

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2>0, \\ 0 & x^2+y^2=0. \end{cases}$$

在坐标原点处沿着哪些方向 f 的方向导数存在?

解. 令 $v = (\cos\theta, \sin\theta)$ 则

$$\frac{\partial f}{\partial v}(0,0) = \lim_{t\to 0}\frac{1}{t}\frac{t\cos\theta t\sin\theta}{|t|} = \lim_{t\to 0}\frac{t}{2|t|}\sin 2\theta.$$

则只有 $\sin 2\theta = 0$ 才有极限存在。解得

$$\theta = 0\text{或}\frac{\pi}{2}\text{或}\pi\text{或}\frac{3\pi}{2}.$$

□

9.1.4 设函数 $f(x,y,z) = |x+y+z|$. 在平面 $x+y+z=0$ 上的每一点处, 沿着哪些方向 f 的方向导数存在?

解. 令 $v = (v_1, v_2, v_3)$, 则在 (x_0, y_0, z_0) 点处

$$\frac{\partial f}{\partial v}(x_0,y_0,z_0) = \lim_{t\to 0}\frac{|tv_1+tv_2+tv_3|}{t},$$

则只有 $v_1+v_2+v_3=0$ 才有极限存在. 故沿着平面内的任意方向都可以. □

9.1.5

(1) 设 $f(x,y) = x+y+\sqrt{x^2+y^2}$. 求 $\frac{\partial f}{\partial x}(0,1), \frac{\partial f}{\partial y}(0,1), \frac{\partial f}{\partial x}(1,2), \frac{\partial f}{\partial y}(1,2)$.

(2) 设 $f(x,y) = \ln(1+xy)+3$. 求 $\frac{\partial f}{\partial x}(1,2), \frac{\partial f}{\partial y}(1,2)$.

(3) 设 $f(x,y) = e^{x+y^2}+\sin x^2y$. 求 $\frac{\partial f}{\partial x}(1,1), \frac{\partial f}{\partial y}(1,1)$.

解. (1)

$$\frac{\partial f}{\partial x} = 1+\frac{x}{\sqrt{x^2+y^2}}, \frac{\partial f}{\partial y} = 1+\frac{y}{\sqrt{x^2+y^2}}$$
$$\Rightarrow \frac{\partial f}{\partial x}(0,0) = 1, \frac{\partial f}{\partial y}(0,0) = 2, \frac{\partial f}{\partial x}(1,2) = 1+\frac{\sqrt{5}}{5}, \frac{\partial f}{\partial y}(1,2) = 1+\frac{2\sqrt{5}}{5}.$$

(2)

$$\frac{\partial f}{\partial x} = \frac{y}{1+xy}, \frac{\partial f}{\partial y} = \frac{x}{1+xy}$$
$$\Rightarrow \frac{\partial f}{\partial x}(1,2) = \frac{2}{3}, \frac{\partial f}{\partial y}(1,2) = \frac{1}{3}.$$

(3)

$$\frac{\partial f}{\partial x} = e^{x+y^2} + 2xy\cos x^2 y, \frac{\partial f}{\partial y} = 2e^{x+y^2} + \cos(x^2 y)x^2$$
$$\Rightarrow \frac{\partial f}{\partial x}(1,1) = e^2 + 2\cos 1, \frac{\partial f}{\partial y}(1,1) = 2e^2 + \cos 1.$$

□

9.1.6 计算偏导数

(2) $z = \tan\frac{x^2}{y}$;

(4) $z = \log(x+y^2)$;

(6) $z = \sin(xy)$;

(8) $u = e^{xyz}$;

(10) $u = \log(x+y^2+z^3)$;

(12) $z = \arcsin(x_1^2 + \cdots + x_n^2)$.

解.

$$\begin{aligned}
&(2) \quad \frac{\partial z}{\partial x} = \frac{2x}{\cos^2(\frac{x^2}{y})y}, \frac{\partial z}{\partial y} = -\frac{x^2}{\cos^2(\frac{x^2}{y})y^2};\\
&(4) \quad \frac{\partial z}{\partial x} = \frac{1}{x+y^2}, \frac{\partial z}{\partial y} = \frac{2y}{x+y^2};\\
&(6) \quad \frac{\partial z}{\partial x} = y\cos(xy), \frac{\partial z}{\partial y} = x\cos(xy);\\
&(8) \quad \frac{\partial u}{\partial x} = yze^{xyz}, \frac{\partial u}{\partial y} = xze^{xyz}, \frac{\partial u}{\partial z} = xye^{xyz};\\
&(10) \quad \frac{\partial u}{\partial x} = \frac{1}{x+y^2+z^3}, \frac{\partial u}{\partial y} = \frac{2y}{x+y^2+z^3}, \frac{\partial u}{\partial z} = \frac{3z^2}{x+y^2+z^3};\\
&(12) \quad \frac{\partial z}{\partial x_i} = \frac{2x_i}{\sqrt{1-(x_1^2+...+x_n^2)^2}}.
\end{aligned}$$

□

4 Week 4

9.2.1 设

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2}, & x^2+y^2>0,\\ 0, & x=y=0. \end{cases}$$

求证：函数 f 在原点处的各个方向导数存在, 但在原点处 f 不可微。

解. 令 $v = (\cos\theta, \sin\theta)$, 则沿着 v 方向的方向导数为

$$\frac{\partial f}{\partial v}(0,0) = \lim_{t\to 0}\frac{1}{t}\cdot\frac{t^2\cos^2\theta\cdot t\sin\theta}{t^4\cos^4\theta + t^2\sin^2\theta} = \lim_{t\to 0}\frac{\cos^2\theta\sin\theta}{t^2\cos^4\theta+\sin^2\theta}$$

若 $\sin\theta \neq 0$ 则有

$$\frac{\partial f}{\partial v}(0,0) = \frac{\cos^2\theta}{\sin\theta};$$

若 $\sin\theta = 0$ 则有

$$\frac{\partial f}{\partial v}(0,0) = 0.$$

故各个方向导数存在.

但取 $x=0$ 方向和 $y=x^2$ 方向逼近原点，分别得到 0 和 $\frac{1}{2}$, 故在原点处不连续，故不可微。 □

9.2.2 求证：函数 $f(x,y)=\sqrt{|xy|}$ 在原点处不可微。

解. 先计算

$$\frac{\partial f}{\partial x}(0,0)=\lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t}=0,$$

类似计算有

$$\frac{\partial f}{\partial y}(0,0)=\lim_{t\to 0}\frac{f(0,t)-f(0,0)}{t}=0,$$

假设 f 在原点处可微, 则只有

$$\lim_{x\to 0,y\to 0}\frac{f(x,y)-\frac{\partial f}{\partial x}(0,0)x-\frac{\partial f}{\partial y}(0,0)y}{\sqrt{x^2+y^2}}=0,$$

于是我们有

$$\lim_{x\to 0,y\to 0}\frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}}=0,$$

但是取 $x=y$ 方向趋向于原点, 则得到极限为 $\frac{1}{\sqrt{2}}$, 矛盾。 □

9.2.4 求下列函数在指定点处的微分:

(2)$f(x,y,z)=\ln(x+y-z)+e^{x+y}\sin z$，在点 $(1,2,3)$ 处;

(4)$u=\sin(x_1+x_x^2+...+x_n^n)$, 在点 $(x_1,x_2,...,x_n)$ 处

解. (2) 分别计算 x,y,z 的偏导数得到

$$\frac{\partial f}{\partial x}=\frac{1}{x+y-z}+e^{x+y}\sin z$$
$$\frac{\partial f}{\partial y}=\frac{1}{x+y-z}+e^{x+y}\sin z$$
$$\frac{\partial f}{\partial z}=-\frac{1}{x+y-z}+e^{x+y}\cos z$$

则带入 $(1,2,1)$ 点处，得到

$$\mathrm{d}f=(1+e^2\sin 1)\mathrm{d}x+(1+e^2\sin 1)\mathrm{d}y+(-1+e^2\cos 1)\mathrm{d}z$$

(4) 分别计算 x_i 的偏导数得到

$$\frac{\partial f}{\partial x_i}=i*\cos(x_1+x_2^2+...+x_n^n)*x_i^{i-1}$$

得到在 $(x_1,x_2,...,x_n)$ 处的微分为

$$\mathrm{d}f=\sum_{i=1}^{n}i*\cos(x_1+x_2^2+...+x_n^n)*x_i^{i-1}\mathrm{d}x_i$$

□

9.2.5 计算下列函数 f 的 Jacobi 矩阵 $\mathbf{J}f$:

(2)$f(x,y,z)=x^2y\sin yz$;

(4)$f(x_1,x_2,...,x_n)=(x_1^2+x_2^2+...+x_n^2)^{\frac{1}{2}}$.

解. (2)

$$\big(2xy\sin(yz), x^2\sin(yz)+x^2yz\cos(yz), x^2y^2\cos(yz)\big).$$

(4)

$$\Big(\frac{x_1}{(x_1^2+x_2^2+...+x_n^2)^{\frac{1}{2}}}, \cdots, \frac{x_n}{(x_1^2+x_2^2+...+x_n^2)^{\frac{1}{2}}}\Big).$$

□

9.2.6 证明：二元函数

$$f(x,y)=\begin{cases} (x^2+y^2)\sin\frac{1}{x^2+y^2}, & (x,y)\neq(0,0), \\ 0, & (x,y)=(0,0) \end{cases}$$

在 $(0,0)$ 处可微，但它的两个偏导数 $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ 在 $(0,0)$ 处不连续.

解. 分别计算 x,y,z 的偏导数得到

$$\frac{\partial f}{\partial x}(0,0)=\lim_{t\to 0}\frac{t^2\sin\frac{1}{t^2}}{t}=0,$$

类似计算有

$$\frac{\partial f}{\partial y}(0,0)=\lim_{t\to 0}\frac{t^2\sin\frac{1}{t^2}}{t}=0.$$

由于

$$\lim_{x\to 0,y\to 0}\frac{f(x,y)-\frac{\partial f}{\partial x}(0,0)x-\frac{\partial f}{\partial y}(0,0)y}{\sqrt{x^2+y^2}}=\lim_{x\to 0,y\to 0}\sqrt{x^2+y^2}\sin\frac{1}{x^2+y^2}=0,$$

故其在原点处可微. 而在不在原点处的偏导数为

$$\frac{\partial f}{\partial x}=2x\sin\frac{1}{x^2+y^2}-\frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}.$$

沿 $y=0$ 方向趋向于原点时，极限不存在。故不连续。
关于 y 的偏导数同理可得不连续。 □

9.3.2 计算下列映射的 Jacobi 矩阵:
(1)$\boldsymbol{f}(r,\theta)=(r\cos\theta, r\sin\theta)$;
(2)$\boldsymbol{f}(r,\theta,z)=(r\cos\theta, r\sin\theta, z)$;
(3)$\boldsymbol{f}(r,\theta,\phi)=(r\cos\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\varphi)$

解. (1)

$$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

(2)

$$\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(3)

$$\begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\sin\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

□

9.3.3 设区域 $D \subset \mathbb{R}^n$, 映射 $\boldsymbol{f}, \boldsymbol{g} : D \to \mathbb{R}^n$. 求证:

(3) 当 $m = 1$ 时，有 $\mathbf{J}(\boldsymbol{f}\boldsymbol{g}) = \boldsymbol{g}\mathbf{J}\boldsymbol{f} + \boldsymbol{f}\mathbf{J}\boldsymbol{g}$;

(4) 当 $m > 1$ 时，有

$$\mathbf{J} < \boldsymbol{f}, \boldsymbol{g} >= \boldsymbol{g}(\mathbf{J}\boldsymbol{f}) + \boldsymbol{f}(\mathbf{J}\boldsymbol{g}),$$

解. (3) 由

$$\frac{\partial \boldsymbol{f}\boldsymbol{g}}{\partial x} = \frac{\partial \boldsymbol{f}}{\partial x} * \boldsymbol{g} + \boldsymbol{f} * \frac{\partial \boldsymbol{g}}{\partial x}$$

这对应于问题中每个分量的值，故成立。(4)

$$\begin{aligned}\mathbf{J}\langle \boldsymbol{f}, \boldsymbol{g}\rangle &= \Big(\frac{\partial \sum_{k=1}^m f_k g_k}{\partial x_1}, ..., \frac{\partial \sum_{k=1}^m f_k g_k}{\partial x_n}\Big) = \Big(\sum_{k=1}^m \frac{\partial f_k g_k}{\partial x_1}, ..., \sum_{k=1}^m \frac{\partial f_k g_k}{\partial x_n}\Big) \\ &= \Big(\sum_{k=1}^m \frac{\partial f_k}{\partial x_1} \cdot g_k + f_k \cdot \frac{\partial g}{\partial x_1}, ..., \sum_{k=1}^m \frac{\partial f_k}{\partial x_n} \cdot g_k + f_k \cdot \frac{\partial g_k}{\partial x_n}\Big) = \boldsymbol{g}(\mathbf{J}\boldsymbol{f}) + \boldsymbol{f}(\mathbf{J}\boldsymbol{g})\end{aligned}$$

□

9.3.6 设映射 $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$. 如果

$$\boldsymbol{f}(\lambda \boldsymbol{x} + \mu \boldsymbol{y}) = \lambda \boldsymbol{f}(\boldsymbol{x}) + \mu \boldsymbol{f}(\boldsymbol{y})$$

对一切 $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ 和一切 $\lambda, \mu \in \mathbb{R}$ 成立, 则称 $\boldsymbol{f}$ 是线性映射。证明:

(1)$\boldsymbol{f}(0) = 0$;

(2)$\boldsymbol{f}(-\boldsymbol{x}) = -\boldsymbol{f}(\boldsymbol{x})$ $(\boldsymbol{x} \in \mathbb{R}^n)$;

(3) 映射 $\boldsymbol{f}$ 由 $\boldsymbol{f}(\mathbf{e}_1), ..., \boldsymbol{f}(\mathbf{e}_n)$ 完全确定。

解. (1) 取 $\lambda = \mu = 0$, 有 $\boldsymbol{f}(0) = 0$.

(2) 取 $\lambda = 1, \mu = -1, \boldsymbol{y} = \boldsymbol{x}$, 有 $\boldsymbol{f}(-\boldsymbol{x}) = -\boldsymbol{f}(\boldsymbol{x})$.

(3)$\forall \boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x} = \boldsymbol{x}_1\mathbf{e}_1 + ... + \boldsymbol{x}_n\mathbf{e}_n$ 则有 $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}_1\boldsymbol{f}(\mathbf{e}_1) + ... + \boldsymbol{x}_n\boldsymbol{f}(\mathbf{e}_n)$ 故映射 $\boldsymbol{f}$ 由 $\boldsymbol{f}(\mathbf{e}_1), ..., \boldsymbol{f}(\mathbf{e}_n)$ 完全确定。 □

9.3.7 设 $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ 为线性映射. 试求 $\mathbf{J}\boldsymbol{f}$.

解.

$$\mathbf{J}\boldsymbol{f} = (\boldsymbol{f}(\mathbf{e}_1), \boldsymbol{f}(\mathbf{e}_2), ..., \boldsymbol{f}(\mathbf{e}_n))$$

□

9.4.2 设 $u = f(xy)$. 证明:

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0.$$

解.

$$\frac{\partial u}{\partial x} = f'(xy)y$$
$$\frac{\partial u}{\partial y} = f'(xy)x$$

代入即恒有

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0.$$

□

9.4.3 设

$$u = f(\log x + \frac{1}{y}),$$

证明:

$$x\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} = 0.$$

解.

$$\frac{\partial u}{\partial x} = f'(\log x + \frac{1}{y})\frac{1}{x}$$
$$\frac{\partial u}{\partial y} = -f'(\log x + \frac{1}{y})\frac{1}{y^2}$$

代入即恒有

$$x\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} = 0.$$

□

9.4.5 求以下 u 的一切偏导数:

(1)$u = f(x+y, xy)$;

(2)$u = f(x, xy, xyz)$;

(3)$u = f(\frac{x}{y}, \frac{y}{z})$.

解. (1)

$$\frac{\partial u}{\partial x} = f_1'(x+y, xy) + yf_2'(x+y, xy)$$
$$\frac{\partial u}{\partial y} = f_1'(x+y, xy) + xf_2'(x+y, xy)$$

(2)

$$\frac{\partial u}{\partial x} = f_1'(x, xy, xyz) + yf_2'(x, xy, xyz) + yzf_3'(x, xy, xyz),$$
$$\frac{\partial u}{\partial y} = xf_2'(x, xy, xyz) + xzf_3'(x, xy, xyz),$$
$$\frac{\partial u}{\partial z} = xyf_3'(x, xy, xyz).$$

(3)

$$\frac{\partial u}{\partial x} = \frac{1}{y}f_1'(\frac{x}{y}, \frac{y}{z}),$$
$$\frac{\partial u}{\partial y} = -\frac{x}{y^2}f_1'(\frac{x}{y}, \frac{y}{z}) + \frac{1}{z}f_2'(\frac{x}{y}, \frac{y}{z}),$$
$$\frac{\partial u}{\partial z} = -\frac{y}{z^2}f_2'(\frac{x}{y}, \frac{y}{z})$$

□

9.4.7 设 $u = x^2y - xy^2$, 且 $x = r\cos\theta$,$y = r\sin\theta$. 求 $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}$.

解.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

计算有

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2xy - y^2, & \frac{\partial u}{\partial y} &= x^2 - 2xy. \\ \frac{\partial x}{\partial r} &= \cos\theta, & \frac{\partial y}{\partial r} &= \sin\theta. \\ \frac{\partial x}{\partial \theta} &= -r\sin\theta, & \frac{\partial y}{\partial \theta} &= r\cos\theta. \end{aligned}$$

代入有

$$\begin{aligned} \frac{\partial u}{\partial r} &= 3r^2(\cos^2\theta\sin\theta - \sin^2\theta\cos\theta), \\ \frac{\partial u}{\partial r} &= r^3(\sin^3\theta - 2\sin^2\theta\cos\theta - 2\sin\theta\cos^2\theta + \cos^3\theta). \end{aligned}$$

□

9.4.8 设 $f(x,y,z) = F(u,v,w)$, 其中 $x^2 = vw, y^2 = wu, z^2 = uv$, 求证:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w}$$

解. 下面都考虑导数存在，故不考虑 $u,v,w = 0$ 的情形由

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}, \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}, \frac{\partial F}{\partial w} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial w},$$

代入要证的式子，则只用证明

$$x = \frac{\partial x}{\partial v}v + \frac{\partial x}{\partial w}w$$

其余两条同理，由于 $x^2 = vw$ 两边分别对 v 和 w 求导

$$2x\frac{\partial x}{\partial v} = w, 2x\frac{\partial x}{\partial w} = v.$$

代入即有等式成立. □

9.4.9 求以下的 $\mathbf{J}(\boldsymbol{f}\circ\boldsymbol{g})$.

(2)$\boldsymbol{f}(x,y) = (\varphi(x+y), \varphi(x-y)), \boldsymbol{g}(s,t) = (e^t, e^{-t})$;

(3)$\boldsymbol{f}(x,y,z) = (x^2+y+z, 2x+y+z^2, 0)$, $\boldsymbol{g}(u,v,w) = (uv^2w^2, w^2\sin v, u^2e^v)$.

解. (2)

$$\mathbf{J}\boldsymbol{f} = \begin{pmatrix} \varphi'(x+y) & \varphi'(x+y) \\ \varphi'(x-y) & -\varphi'(x-y) \end{pmatrix}, \mathbf{J}\boldsymbol{g} = \begin{pmatrix} 0 & e^t \\ 0 & -e^{-t} \end{pmatrix},$$

由链式法则

$$\mathbf{J}(\boldsymbol{f}\circ\boldsymbol{g}) = \mathbf{J}\boldsymbol{f}\mathbf{J}\boldsymbol{g} = \begin{pmatrix} 0 & \varphi'(e^t+e^{-t})*e^t - \varphi'(e^t+e^{-t})*e^{-t} \\ 0 & \varphi'(e^t-e^{-t})*e^t + \varphi'(e^t-e^{-t})*e^{-t} \end{pmatrix}.$$

(3)

$$\mathbf{J}\boldsymbol{f} = \begin{pmatrix} 2x & 1 & 1 \\ x & 1 & 2z \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{J}\boldsymbol{g} = \begin{pmatrix} v^2w^2 & 2uvw^2 & 2uv^2w \\ 0 & w^2\cos v & 2w\sin v \\ 2ue^v & u^2e^v & 0 \end{pmatrix},$$

由链式法则

$$\mathbf{J}(\boldsymbol{f}\circ\boldsymbol{g})=\mathbf{J}\boldsymbol{f}\mathbf{J}\boldsymbol{g}=\begin{pmatrix}2uv^4w^4+2ue^v & 4u^2v^3w^4+w^2\cos v+u^2e^v & 4u^2v^4w^3+2w\sin v\\ 2v^2w^2+4u^3e^{2v} & 4uvw^2+w^2\cos v+2u^4e^{2v} & 4uv^2w+2w\sin v\\ 0 & 0 & 0\end{pmatrix}.$$

□

9.4.10 设函数 $f(x,y,z)$ 在 $\mathbb{R}^3$ 中可微，又设 $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ 是 $\mathbb{R}^3$ 中三个互相垂直的方向. 求证:

$$\left(\frac{\partial f}{\partial \mathbf{e}_1}\right)^2+\left(\frac{\partial f}{\partial \mathbf{e}_2}\right)^2+\left(\frac{\partial f}{\partial \mathbf{e}_3}\right)^2=\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2$$

解. 记

$$A=(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$$

则由方向导数计算方法得到

$$\begin{pmatrix}\frac{\partial f}{\partial \mathbf{e}_1}\\ \frac{\partial f}{\partial \mathbf{e}_2}\\ \frac{\partial f}{\partial \mathbf{e}_3}\end{pmatrix}=A^T\begin{pmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\\ \frac{\partial f}{\partial z}\end{pmatrix}$$

由 A 为正交阵带入得

$$\left(\frac{\partial f}{\partial \mathbf{e}_1}\right)^2+\left(\frac{\partial f}{\partial \mathbf{e}_2}\right)^2+\left(\frac{\partial f}{\partial \mathbf{e}_3}\right)^2=\begin{pmatrix}\frac{\partial f}{\partial \mathbf{e}_1} & \frac{\partial f}{\partial \mathbf{e}_2} & \frac{\partial f}{\partial \mathbf{e}_3}\end{pmatrix}\begin{pmatrix}\frac{\partial f}{\partial \mathbf{e}_1}\\ \frac{\partial f}{\partial \mathbf{e}_2}\\ \frac{\partial f}{\partial \mathbf{e}_3}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\end{pmatrix}AA^T\begin{pmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\\ \frac{\partial f}{\partial Z}\end{pmatrix}=\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2$$

□

9.5.2 设参数曲线

$$\boldsymbol{r}(t)=\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2},1\right)$$

求证：在任意 $t\in\mathbb{R}$, 径向量 $\boldsymbol{r}(t)$ 与切向量 $\boldsymbol{r}'(t)$ 互相正交. 问这是一条什么曲线?

解.

$$\boldsymbol{r}'(t)=\left(\frac{2-2t^2}{(1+t^2)^2},\frac{-4t}{(1+t^2)^2},0\right)$$

于是我们有

$$\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)=0$$

故正交. 计算可得 $||\boldsymbol{r}||=1$, 且这条曲线落在 xy 平面内, 故为圆. □

9.5.3 讨论平面曲线

$$\boldsymbol{r}(t)=(e^t\cos t,e^t\sin t)$$

求证: 在曲线上的每一点处，切向量和径向量交成定角 $\frac{\pi}{4}$.

解.

$$\boldsymbol{r}'(t)=(e^t(\cos t-\sin t),e^t(\sin t+\cos t))$$

于是我们有

$$\cos\theta=\frac{\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)}{||\boldsymbol{r}(t)||\cdot||\boldsymbol{r}'(t)||}=\frac{1}{\sqrt{2}}$$

故切向量和径向量交成定角 $\frac{\pi}{4}$. □

9.5.5 证明:

(1)$(\lambda(t)\boldsymbol{a}(t))'=\lambda'(t)\boldsymbol{a}(t)+\lambda(t)\boldsymbol{a}'(t)$;

(2)$(\boldsymbol{b}(t)\cdot\boldsymbol{a}(t))'=\boldsymbol{b}'(t)\boldsymbol{a}(t)+\boldsymbol{b}(t)\boldsymbol{a}'(t)$;

(3)$(\boldsymbol{b}(t)\times\boldsymbol{a}(t))'=\boldsymbol{b}'(t)\times\boldsymbol{a}(t)+\boldsymbol{b}(t)\times\boldsymbol{a}'(t)$

解. (1)

$$\begin{aligned}(\lambda(t)\boldsymbol{a}(t))'&=(\lambda(t)\boldsymbol{a}_1(t),\lambda(t)\boldsymbol{a}_2(t),...,\lambda(t)\boldsymbol{a}_n(t))'\\&=(\lambda'(t)\boldsymbol{a}_1(t)+\lambda(t)\boldsymbol{a}_1'(t),\lambda'(t)\boldsymbol{a}_2(t)+\lambda(t)\boldsymbol{a}_2'(t),...,\lambda'(t)\boldsymbol{a}_n(t)+\lambda(t)\boldsymbol{a}_n'(t),)\\&=\lambda'(t)\boldsymbol{a}(t)+\lambda(t)\boldsymbol{a}'(t)\end{aligned}$$

(2)

$$\begin{aligned}(\boldsymbol{b}(t)\cdot\boldsymbol{a}(t))'&=(\boldsymbol{b}_1(t)\boldsymbol{a}_1(t)+\boldsymbol{b}_2(t)\boldsymbol{a}_2(t)+\cdots+b_n(t)a_n(t))'\\&=\boldsymbol{b}_1'(t)\boldsymbol{a}_1(t)+\boldsymbol{b}_1(t)\boldsymbol{a}_1'(t)+\boldsymbol{b}_2'(t)\boldsymbol{a}_2(t)+\boldsymbol{b}_2(t)\boldsymbol{a}_2'(t)+\cdots+\boldsymbol{b}_n'(t)\boldsymbol{a}_n(t)+\boldsymbol{b}_n(t)\boldsymbol{a}_n'(t)\\&=\boldsymbol{b}'(t)\boldsymbol{a}(t)+\boldsymbol{b}(t)\boldsymbol{a}'(t)\end{aligned}$$

(3) 考虑 $n=3$ 的情形

$$\begin{aligned}(\boldsymbol{b}(t)\times\boldsymbol{a}(t))'&=(\boldsymbol{b}_2(t)\boldsymbol{a}_3(t)-\boldsymbol{b}_3(t)\boldsymbol{a}_2(t),\boldsymbol{b}_3(t)\boldsymbol{a}_1(t)-\boldsymbol{b}_1(t)\boldsymbol{a}_3(t),\boldsymbol{b}_1(t)\boldsymbol{a}_2(t)-\boldsymbol{b}_2(t)\boldsymbol{a}_1(t))'\\&=(\boldsymbol{b}_2'(t)\boldsymbol{a}_3(t)-\boldsymbol{b}_3'(t)\boldsymbol{a}_2(t)+\boldsymbol{b}_2(t)\boldsymbol{a}_3'(t)-\boldsymbol{b}_3(t)\boldsymbol{a}_2'(t),\\&\boldsymbol{b}_3'(t)\boldsymbol{a}_1(t)-\boldsymbol{b}_1'(t)\boldsymbol{a}_3(t)+\boldsymbol{b}_3(t)\boldsymbol{a}_1'(t)-\boldsymbol{b}_1(t)\boldsymbol{a}_3'(t),\boldsymbol{b}_1'(t)\boldsymbol{a}_2(t)-\boldsymbol{b}_2'(t)\boldsymbol{a}_1(t)+\boldsymbol{b}_1(t)\boldsymbol{a}_2'(t)-\boldsymbol{b}_2(t)\boldsymbol{a}_1'(t))\\&=\boldsymbol{b}'(t)\times\boldsymbol{a}(t)+\boldsymbol{b}(t)\times\boldsymbol{a}'(t).\end{aligned}$$

□

9.5.7 讨论椭圆

$$\begin{cases}x=a\cos t\\y=b\sin t\end{cases},(0\le t\le 2\pi).$$

(1) 求椭圆在每一点处的切向量;

(2) 证明椭圆的光学性质.

解. (1)

$$\boldsymbol{r}'(t)=(-a\sin t,b\cos t)$$

(2) 椭圆的焦点为 $F_1(-c,0)$, $F_2(c,0)$. 椭圆在 $P=\boldsymbol{x}(t)$ 点处的切向量为

$$\boldsymbol{n}=(b\cos t,a\sin t)$$

证明夹角相等只需证明

$$\frac{\overrightarrow{F_1P}\cdot \boldsymbol{n}}{||\overrightarrow{F_1P}||} = \frac{\overrightarrow{F_2P}\cdot \boldsymbol{n}}{||\overrightarrow{F_2P}||},$$

代入 $\overrightarrow{F_1P} = (a\cos t + c, b\sin t)$,$\overrightarrow{F_2P} = (a\cos t - c, b\sin t)$ 即可得到等式成立. □

9.5.8 求下列曲线的曲率:

(2)$\boldsymbol{r}(t) = (a(3t-t^2), 3at^2, a(3t+t^3))$, 其中常数 $a>0$.

解. 分别计算 $\boldsymbol{r}'(t)$ 和 $\boldsymbol{r}''(t)$ 有

$$\boldsymbol{r}'(t) = (3a(1-t^2), 6at, 3a(1+t^2)), \boldsymbol{r}''(t) = (-6at, 6a, 6at),$$

则有

$$\boldsymbol{r}'(t)\times\boldsymbol{r}''(t) = (18a^2(t^2-1), -36a^2t, 18a^2(t^2+1)),$$

代入曲率的计算公式有

$$\begin{aligned} k(t) &= \frac{||\boldsymbol{r}^{'}(t)\times\boldsymbol{r}^{''}(t)||}{||\boldsymbol{r}^{'}(t)||^3} \\ &= \frac{1}{3a(3t^2+1)}. \end{aligned}$$

□

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9.5.10 求法向量和切平面方程。

(3) $3x^2+2y^2-2z-1=0$ 在 $(1,1,2)$ 处;

(4) $z = y + \log(x/z)$ 在 $(1,1,1)$ 处.

解. (3)$F(x,y,z) = 3x^2+2y^2-2z-1$, 故其偏导

$$\frac{\partial F}{\partial x} = 6x, \frac{\partial F}{\partial y} = 4y, \frac{\partial F}{\partial z} = -2.$$

代入点的坐标得

$$\mathbf{J}F(\boldsymbol{p}_0) = (6,4,-2)$$

故单位法向量为

$$\boldsymbol{n} = \frac{1}{\sqrt{15}}(3,2,-1),$$

切平面为 $3(x-1)+2(y-1)-(z-2)=0$, 化简得

$$3x+2y-z-3=0.$$

(4)$F(x,y,z) = y+\log(x/z)$, 故其偏导

$$\frac{\partial F}{\partial x} = \frac{1}{x}, \frac{\partial F}{\partial y} = 1, \frac{\partial F}{\partial z} = -\frac{1}{z} - 1.$$

代入点的坐标得

$$\mathbf{J}F(\boldsymbol{p}_0) = (1,1,-2)$$

故单位法向量为

$$\boldsymbol{n}=\frac{1}{\sqrt{6}}(1,1,-2),$$

切平面为 $(x-1)+(y-1)-2(z-1)=0$, 化简得

$$x+y-2z=0.$$

□

9.5.11 求 $x^2+2y^2+3z^2=21$ 上所有平行于 $x+4y+6z=0$ 的切平面。

解. $\mathbf{J}F(x,y,z)=(2x,4y,6z)$, 在 (x_0,y_0,z_0) 处的切平面方程为

$$x_0(x-x_0)+2y_0(y-y_0)+3z_0(z-z_0)=0.$$

故得到方程组

$$\begin{cases}x_0:2y_0:3z_0=1:4:6\\ x_0^2+2y_0^2+3z_0^2=21\end{cases}$$

解得

$$(x_0,y_0,z_0)=\pm(1,2,2)$$

得到2 个满足条件得切平面 $(x-1)+4(y-2)+6(z-2)=0$ 和 $-(x+1)-4(y+2)-6(z+2)=0$, 化简得

$$x+4y+6z\pm21=0.$$

□

9.5.12 曲面 $z=xe^{x/y}$ 上所有切平面都通过原点。

解.

$$F=xe^{x/y}-z,\boldsymbol{J}F=(e^{x/y}+\frac{x}{y}e^{x/y},-\frac{x^2}{y^2}e^{x/y},-1),$$

在 (x_0,y_0,z_0) 处的切平面方程为

$$\left(e^{x_0/y_0}+\frac{x_0}{y_0}e^{x_0/y_0}\right)(x-x_0)-\frac{x_0^2}{y_0^2}e^{x_0/y_0}(y-y_0)-(z-z_0)=0,$$

代入 $x=y=z=0$ 得到等式左边 $=-x_0e^{x_0/y_0}-\frac{x_0^2}{y_0}e^{x_0/y_0}+\frac{x_0^2}{y_0}e^{x_0/y_0}+z_0=-x_0e^{x_0/y_0}+z_0=0=$ 右边。 □

9.5.12 试给出正数 $\lambda>0$, 使曲面 $xyz=\lambda$ 与 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 在某一点相切（有共同的切平面）。

解. 设

$$F_1(x,y,z)=xyz-\lambda,F_2(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1,$$

$$\boldsymbol{J}F_1=(yz,xz,xy)=\left(\frac{\lambda}{x},\frac{\lambda}{y},\frac{\lambda}{z}\right),\boldsymbol{J}F_2=\left(\frac{2x}{a^2},\frac{2y}{b^2},\frac{2z}{c^2}\right).$$

由相切可得 $\boldsymbol{J}F_1$ 与 $\boldsymbol{J}F_2$ 平行，得到

$$k\Big(\frac{\lambda}{x},\frac{\lambda}{y},\frac{\lambda}{z}\Big)=\Big(\frac{2x}{a^2},\frac{2y}{b^2},\frac{2z}{c^2}\Big)\Rightarrow\frac{k\lambda}{2}=\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2},$$

代入 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 得到

$$\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2}=\frac{1}{3}$$

结合$\lambda>0$得到

$$\lambda=xyz=\sqrt{\frac{a^2}{3}\cdot\frac{b^2}{3}\cdot\frac{c^2}{3}}=\frac{abc}{3\sqrt{3}}.$$

□

9.5.16 求曲面 $x^2+y^2+z^2=x$ 的切平面，使其垂直于平面 $x-y-z=2$ 和 $x-y-z/2=2$.

解. $F(x,y,z)=x^2+y^2+z^2-x,\mathbf{J}F=(2x-1,2y,2z)$. 由于切平面垂直于平面 $x-y-z=2$ 和 $x-y-z/2=2$, 得

$$\begin{cases}(2x-1,2y,2z)\perp(1,-1,-1)\\(2x-1,2y,2z)\perp(1,-1,-\frac{1}{2})\end{cases}\Rightarrow\begin{cases}2x-1-2y-2z=0\\2x-1-2y-z=0\end{cases}\Rightarrow\begin{cases}y=\frac{2x-1}{2}\\z=0\end{cases}\Rightarrow\boldsymbol{n}=\frac{1}{\sqrt{2}}(1,1,0)$$

为切平面的单位法向量，再求切平面与原曲面的交点。代入 $y=\frac{2x-1}{2}$ 和 $z=0$,

$$\begin{cases}x^2+y^2-x=0\\2y=2x-1\end{cases}\Rightarrow\begin{cases}x=\frac{2+\sqrt{2}}{4}\\y=\frac{\sqrt{2}}{4}\end{cases}\text{ or }\begin{cases}x=\frac{2-\sqrt{2}}{4}\\y=-\frac{\sqrt{2}}{4}\end{cases},$$

代入并化简得满足条件的切平面为

$$x+y-\frac{1+\sqrt{2}}{2}=0\text{ 和 }x+y-\frac{1-\sqrt{2}}{2}=0.$$

□

9.5.22 求 E,F,G.

(1) 椭球面：$\boldsymbol{r}(u,v)=(a\sin u\cos v,b\sin u\sin v,c\cos u)$;

(2) 单叶双曲面：$\boldsymbol{r}(u,v)=(a\cosh u\cos v,b\cosh u\sin v,c\sinh u)$;

(3) 椭圆抛物线：$\big(u,v,\frac{1}{2}\big(\frac{u^2}{a^2}+\frac{v^2}{b^2}\big)\big)$.

解.

(1) $\boldsymbol{r}_u = (a\cos u\cos v, b\cos u\sin v, -c\sin u), \boldsymbol{r}_v = (-a\sin u\sin v, b\sin u\cos v, 0)$;

$E = ||\boldsymbol{r}_u||^2 = a^2\cos^2 u\cos^2 v + b^2\cos^2 u\sin^2 v + c^2\sin^2 u$;

$F = \langle \boldsymbol{r}_u, \boldsymbol{r}_v\rangle = (b^2 - a^2)\cos u\sin u\cos v\sin v$;

$G = ||\boldsymbol{r}_v||^2 = a^2\sin^2 u\sin^2 v + b^2\sin^2 u\cos^2 v$.

(2) $\boldsymbol{r}_u = (a\sinh u\cos v, b\sinh u\sin v, c\cosh u), \boldsymbol{r}_v = (-a\cosh u\sin v, b\cosh u\cos v, 0)$;

$E = ||\boldsymbol{r}_u||^2 = a^2\sinh^2 u\cos^2 v + b^2\sinh^2 u\sin^2 v + c^2\cosh^2 u$;

$F = \langle \boldsymbol{r}_u, \boldsymbol{r}_v\rangle = (b^2 - a^2)\sinh u\cosh u\sin v\cos v$;

$G = ||\boldsymbol{r}_v||^2 = a^2\cosh^2 u\sin^2 v + b^2\cosh^2 u\cos^2 v$.

(3) $\boldsymbol{r}_u = (1, 0, \frac{u}{a^2}), \boldsymbol{r}_v = (0, 1, \frac{v}{b^2})$;

$E = ||\boldsymbol{r}_u||^2 = 1 + \frac{u^2}{a^4}$;

$F = \langle \boldsymbol{r}_u, \boldsymbol{r}_v\rangle = \frac{uv}{a^2b^2}$;

$G = ||\boldsymbol{r}_v||^2 = 1 + \frac{v^2}{b^4}$.

□

9.5.23 $I := E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2$, 证明：$I = \mathrm{d}\boldsymbol{r}^2$, 其中 $\boldsymbol{r} = \boldsymbol{r}(u, v)$.

解.

$$\begin{aligned}
\mathrm{d}\boldsymbol{r} &= \boldsymbol{r}_u\mathrm{d}u + \boldsymbol{r}_v\mathrm{d}v \\
\mathrm{d}\boldsymbol{r}^2 &= \langle \boldsymbol{r}_u\mathrm{d}u + \boldsymbol{r}_v\mathrm{d}v, \boldsymbol{r}_u\mathrm{d}u + \boldsymbol{r}_v\mathrm{d}v\rangle \\
&= ||\boldsymbol{r}_u||^2\mathrm{d}u^2 + 2\boldsymbol{r}_u\boldsymbol{r}_v\mathrm{d}u\mathrm{d}v + ||\boldsymbol{r}_v||^2\mathrm{d}v^2 \\
&= E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2 = I.
\end{aligned}$$

□

9.5.24 已知曲面 $I = \mathrm{d}u^2 + (u^2 + a^2)\mathrm{d}v^2$, 求曲面上的曲线 $u = v$ 从 v_1 到 v_2 的弧长，其中 $v_2 > v_1$.

解. 由上题 $\mathrm{d}\boldsymbol{r}^2 = I = \mathrm{d}u^2 + (u^2 + a^2)\mathrm{d}v^2$, 在曲线 $u = v$ 上 $\mathrm{d}u = \mathrm{d}v$, 所以

$$\mathrm{d}\boldsymbol{r}^2 = (1 + a^2 + v^2)\mathrm{d}v^2 \Rightarrow |\mathrm{d}\boldsymbol{r}| = \sqrt{1 + a^2 + v^2}|\mathrm{d}v|,$$

在 v_1 到 v_2 上积分得弧长

$$s = \int_{v_1}^{v_2} |\mathrm{d}\boldsymbol{r}| = \int_{v_1}^{v_2} \sqrt{1 + a^2 + v^2}\mathrm{d}v = \frac{1}{2}v\sqrt{1 + a^2 + v^2} + \frac{1}{2}(1 + a^2)\log(v + \sqrt{1 + a^2 + v^2})\Big|_{v_1}^{v_2}.$$

□

9.6.1 计算 $\frac{\mathrm{d}y}{\mathrm{d}x}$.

(2) $xy - \log y = 0$ 在 $(0, 1)$ 处；

(4) $x^y = y^x$.

解. (2) 隐函数定理:

$$
\begin{aligned}
&F(x,y) = xy - \log y \\
&\frac{\partial F}{\partial x} = y, \frac{\partial F}{\partial y} = x - \frac{1}{y}, \\
&\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{y^2}{xy-1}.
\end{aligned}
$$

在 $(0,1)$ 处有

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

或直接对等式两边求微分得

$$
\begin{aligned}
x\mathrm{d}y + y\mathrm{d}x - \frac{1}{y}\mathrm{d}y = 0 \\
(x - \frac{1}{y})\mathrm{d}y = -y\mathrm{d}x \\
\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y^2}{xy-1}
\end{aligned}
$$

(4) 两边取对数得

$$
\begin{aligned}
y\log x &= x\log y \\
\frac{y}{x}\mathrm{d}x + \log x\mathrm{d}y &= \frac{x}{y}\mathrm{d}y + \log y\mathrm{d}x \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y/x - \log y}{x/y - \log x} = \frac{y^2 - x\log y}{x^2 - y\log x}
\end{aligned}
$$

注：不取对数直接算出的结果

$$\frac{yx^{y-1} - y^x\log y}{xy^{x-1} - x^y\log x}$$

可通过等式 $x^y = y^x$ 化简后证明与上述结果一致。 □

9.6.2 计算 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

(2) $\frac{x}{z} = \log\frac{z}{y}$;

(3) $(x+y+z)e^{-(x+y+z)}$;

(4) $z^2y - xz^3 - 1 = 0$, 在 $(1,2,1)$ 处;

解. (2)

$$
\begin{aligned}
\frac{\partial}{\partial x}: \quad & \frac{1}{z} - \frac{x}{z^2}\frac{\partial z}{\partial x} = \frac{1}{z}\frac{\partial z}{\partial x} & \Rightarrow \frac{\partial z}{\partial x} = \frac{z}{z+x}; \\
\frac{\partial}{\partial y}: \quad & -\frac{x}{z^2}\frac{\partial z}{\partial y} = \frac{1}{z}\frac{\partial z}{\partial y} - \frac{1}{y} & \Rightarrow \frac{\partial z}{\partial y} = \frac{z^2}{y(x+z)}.
\end{aligned}
$$

(3)

$$
\begin{aligned}
\frac{\partial}{\partial x}: \quad & 1 + \frac{\partial z}{\partial x} = -e^{-(x+y+z)}(1 + \frac{\partial z}{\partial x}) & \Rightarrow \frac{\partial z}{\partial x} = -1; \\
\frac{\partial}{\partial y}: \quad & 1 + \frac{\partial z}{\partial y} = -e^{-(x+y+z)}(1 + \frac{\partial z}{\partial y}) & \Rightarrow \frac{\partial z}{\partial y} = -1.
\end{aligned}
$$

(4)

$$
\begin{aligned}
\frac{\partial}{\partial x}: \quad & 2yz\frac{\partial z}{\partial x} - z^3 - 3xz^2\frac{\partial z}{\partial x} = 0 & \Rightarrow \frac{\partial z}{\partial x} = \frac{z^2}{2y - 3xz}; \\
\frac{\partial}{\partial y}: \quad & z^2 + 2yz\frac{\partial z}{\partial y} - 3xz^2\frac{\partial z}{\partial y} = 0 & \Rightarrow \frac{\partial z}{\partial y} = \frac{z}{2xz - 2y}.
\end{aligned}
$$

代入点 $(1,2,1)$得
$$\frac{\partial z}{\partial x}=1,\frac{\partial z}{\partial y}=-1.$$
□

9.6.3 设 $F(x,y,z)=0$, 求证:
$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x}=-1.$$

解. 将 x 看作关于 y,z 的函数 $x(y,z)$, 对 $F(x,y,z)=0$ 两边求关于 y 的偏导得
$$F_x\frac{\partial x}{\partial y}+F_y=0$$
解得
$$\frac{\partial x}{\partial y}=-\frac{F_y}{F_x};$$
同理有
$$\frac{\partial y}{\partial z}=-\frac{F_z}{F_y};$$
$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}.$$
代入即得
$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x}=-1.$$
□

9.6.4 设 $F(x-y,y-z,z-x)=0$, 计算 $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$.

解. 将 z 看作关于 x,y 的函数 $z(x,y)$, 对等式两边分别求关于 x,y 的偏导,
$$\begin{cases} F_1-F_2\frac{\partial z}{\partial x}+F_3\big(\frac{\partial z}{\partial x}-1\big)=0 \\ -F_1+F_2\big(1-\frac{\partial z}{\partial y}\big)+F_3\frac{\partial z}{\partial y}=0 \end{cases}\Rightarrow\begin{cases} \frac{\partial z}{\partial x}=\frac{F_3-F_1}{F_3-F_2} \\ \frac{\partial z}{\partial y}=\frac{F_1-F_2}{F_3-F_2} \end{cases}$$
注：以上所有关于 F 的求导均在 $(x-y,y-z,z-x)$ 处，F_i 表示 F 对第 i 个分量求导。 □

9.6.5 设 $F(x+y+z,x^2+y^2+z^2)=0$, 计算 $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$.

解. 将 z 看作关于 x,y 的函数 $z(x,y)$, 对等式两边分别求关于 x,y 的偏导,
$$\begin{cases} F_1\big(1+\frac{\partial z}{\partial x}\big)+F_2\big(2x+2z\frac{\partial z}{\partial x}\big)=0 \\ F_1\big(1+\frac{\partial z}{\partial y}\big)+F_2\big(2y+2z\frac{\partial z}{\partial u}\big)=0 \end{cases}\Rightarrow\begin{cases} \frac{\partial z}{\partial x}=-\frac{F_1+2xF_2}{F_1+2zF_2} \\ \frac{\partial z}{\partial y}=-\frac{F_1+2yF_2}{F_1+2zF_2} \end{cases}$$
注：以上所有关于 F 的求导均在 $(x+y+z,x^2+y^2+z^2)$ 处，F_i 表示 F 对第 i 个分量求导。 □

6 Week 6

9.7.1 对方程

$$\begin{cases} x^2+y^2+z^2=1, \\ x+y+z=0, \end{cases}$$

计算 $\frac{dy}{dx}$ 和 $\frac{dz}{dx}$, 并作出解释.

解. 记 $F_1(x,y,z)=x^2+y^2+z^2-1$,$F_2(x,y,z)=x+y+z$. 由隐函数定理得 $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}=0$ 确定了 y,z 关于 x 的函数 $(y,z)=\boldsymbol{f}(x)$. 且有

$$\begin{aligned} \mathbf{J}\boldsymbol{f}(x)=\begin{pmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{pmatrix} &= -\begin{pmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial x} \end{pmatrix} \\ &= -\begin{pmatrix} 2y & 2z \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2x \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{x-z}{y-z} \\ -\frac{x-y}{z-y} \end{pmatrix}. \end{aligned}$$

解释：这个曲线为一个圆。 □

9.7.3 对方程

$$\begin{cases} x=t+\frac{1}{t}, \\ y=t^2+\frac{1}{t^2}, \\ z=t^3+\frac{1}{t^3}, \end{cases}$$

计算 $\frac{dy}{dx}$ 和 $\frac{dz}{dx}$.

解. 带入 x,y,z 的方程有 $y=x^2-2$ 和 $z=x^3-3x$ 故求得

$$\frac{dy}{dx}=2x,$$
$$\frac{dz}{dx}=3x^2-3.$$

□

9.7.4 对下列方程, 计算 Jacobi 矩阵

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

(1) $xu-yv=0, yu+xv=1$;

(2) $x+y=u+v, \frac{x}{y}=\frac{\sin u}{\sin v}$.

解. (1) 记 $F_1(x,y,u,v)=xu-yv$,$F_2(x,y,u,v)=yu+xv-1$. 由隐映射定理得 $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}=0$ 确定了 x,y 关于 u,v 的函数 $(x,y)=(x(u,v),y(u,v))$. 且有

$$\begin{aligned}\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} &= -\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix} \\ &= -\begin{pmatrix} u & -v \\ v & u \end{pmatrix}^{-1} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \\ &= -\frac{1}{u^2+v^2}\begin{pmatrix} xu+yv & -yu+xv \\ yu-xv & xu+yv \end{pmatrix}.\end{aligned}$$

(2) 记 $F_1(x,y,u,v)=x+y-u-v$,$F_2(x,y,u,v)=x\sin v-y\sin u$. 由隐映射定理得 $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}=0$ 确定了 x,y 关于 u,v 的函数 $(x,y)=(x(u,v),y(u,v))$. 且有

$$\begin{aligned}\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} &= -\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix} \\ &= -\begin{pmatrix} 1 & 1 \\ \sin v & -\sin u \end{pmatrix}^{-1} \begin{pmatrix} -1 & -1 \\ -y\cos u & x\cos v \end{pmatrix} \\ &= \frac{1}{\sin u+\sin v}\begin{pmatrix} y\cos u+\sin u & -x\cos v+\sin u \\ -y\cos u+\sin v & x\cos v+\sin v \end{pmatrix}.\end{aligned}$$

□

9.7.6 设 $u=f(x,y,z,t),g(y,z,t)=0,h(z,t)=0$, 计算 $\frac{\partial u}{\partial x}$ 和 $\frac{\partial u}{\partial y}$.

解. 记 $F_1(y,z,t)=g(y,z,t)$,$F_2(z,t)=h(z,t)$. 由隐映射定理得 $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}=0$ 确定了 z,t 关于 y 的函数 $(z,t)=\boldsymbol{f}(y)$. 且有

$$\begin{aligned}\begin{pmatrix} \frac{\partial z}{\partial y} \\ \frac{\partial t}{\partial y} \end{pmatrix} &= -\begin{pmatrix} \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial t} \\ \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial t} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial y} \end{pmatrix} \\ &= -\begin{pmatrix} g_2' & g_3' \\ h_1' & h_2' \end{pmatrix}^{-1} \begin{pmatrix} g_1' \\ 0 \end{pmatrix} \\ &= -\frac{1}{g_2'h_2'-g_3'h_1'}\begin{pmatrix} g_1'h_2' \\ -g_1'h_1' \end{pmatrix}.\end{aligned}$$

由链式法则得

$$\frac{\partial u}{\partial x}=f_1'$$

$$\begin{aligned}\frac{\partial u}{\partial y} &= f_2'+f_3'\frac{\partial z}{\partial y}+f_4'\frac{\partial t}{\partial y} \\ &= f_2'-\frac{g_1'(f_4'h_1'-f_3'h_2')}{g_2'h_2'-g_3'h_1'}\end{aligned}$$

□

9.8.1 设 $D \subset \mathbb{R}^n$, 映射 $\boldsymbol{f}: D \to \mathbb{R}^n$. 如果 $\boldsymbol{f}$ 把开集映为开集, 则称 $\mathbf{J}$ 为一个开映射. 问下列映射是不是开映射:

(1) $\boldsymbol{f}(x,y) = \left(x^2, \frac{y}{x}\right)$;

(2) $\boldsymbol{f}(x,y) = (e^x \cos y, e^x \sin y)$;

(3) $\boldsymbol{f}(x,y) = x+y, 2xy^2$.

解. (1)

$$\mathbf{J}\boldsymbol{f}(x,y) = \begin{pmatrix} 2x & 0 \\ -\frac{y}{x^2} & \frac{1}{x} \end{pmatrix},$$

则计算得 $det\mathbf{J}\boldsymbol{f}(x,y) = 2 \neq 0$, 于是 f 为开映射.

(2)

$$\mathbf{J}\boldsymbol{f}(x,y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix},$$

则计算得 $det\mathbf{J}\boldsymbol{f}(x,y) = e^{2x} \neq 0$, 于是 f 为开映射.

(3)

$$\mathbf{J}\boldsymbol{f}(x,y) = \begin{pmatrix} 1 & 1 \\ 2y^2 & 4xy \end{pmatrix},$$

则计算得 $det\mathbf{J}\boldsymbol{f}(x,y) = 2y(2x-y)$, $y \neq 0$ $x \neq 2x$ 时, 有 $det\mathbf{J}\boldsymbol{f}(x,y) \neq 0$, 于是 f 为开映射. □

9.8.2 对第题中的三个映射 $\boldsymbol{f}$, 计算 $\mathbf{J}\boldsymbol{f}^{-1}$.

解. (1)

$$\mathbf{J}\boldsymbol{f}(x,y)^{-1} = \begin{pmatrix} \frac{1}{2x} & 0 \\ \frac{y}{2x^2} & x \end{pmatrix},$$

(2)

$$\mathbf{J}\boldsymbol{f}(x,y)^{-1} = \begin{pmatrix} e^{-x} \cos y & e^{-x} \sin y \\ -e^{-x} \sin y & e^{-x} \cos y \end{pmatrix},$$

(3)

$$\mathbf{J}\boldsymbol{f}(x,y)^{-1} = \frac{1}{2y(2x-y)} \begin{pmatrix} 4xy & -1 \\ -2y^2 & 1 \end{pmatrix},$$

□

9.9.1 求下列函数的二阶偏导数:

(2) $z = \tan \frac{x^2}{y}$;

(4) $z = \arctan \frac{y}{x}$;

(6) $u = xy + yz + zx$;

(8) $u = x^{yz}$;

(10) $u = \arcsin(x_1^2 + x_2^2 + ... + x_n^2)$.

解. (2)

$$\frac{\partial^2 z}{\partial x^2} = \frac{2}{y^2\cos^2\frac{x^2}{y}}(y + 4x^2\tan\frac{x^2}{y}),$$
$$\frac{\partial^2 z}{\partial x\partial y} = \frac{2x}{y^2\cos^2\frac{x^2}{y}}(1 + \frac{2x^2}{y}\tan\frac{x^2}{y}),$$
$$\frac{\partial^2 z}{\partial y^2} = \frac{2x^2}{y^4\cos^2\frac{x^2}{y}}(y + x^2\tan\frac{x^2}{y}).$$

(4)

$$\frac{\partial^2 z}{\partial x^2} = -\frac{2xy}{x^2+y^2},$$
$$\frac{\partial^2 z}{\partial x\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2},$$
$$\frac{\partial^2 z}{\partial y^2} = \frac{-2xy}{x^2+y^2}.$$

(6)

$$\frac{\partial^2 u}{\partial x^2} = 0,$$
$$\frac{\partial^2 u}{\partial y^2} = 0,$$
$$\frac{\partial^2 u}{\partial z^2} = 0,$$
$$\frac{\partial^2 u}{\partial x\partial y} = 0,$$
$$\frac{\partial^2 u}{\partial x\partial z} = 0,$$
$$\frac{\partial^2 u}{\partial y\partial z} = 0,$$

(8)

$$\frac{\partial^2 u}{\partial x^2} = yz(yz-1)x^{yz-2},$$
$$\frac{\partial^2 u}{\partial y^2} = z^2x^{yz}(\ln x)^2,$$
$$\frac{\partial^2 u}{\partial z^2} = y^2x^{yz}(\ln x)^2,$$
$$\frac{\partial^2 u}{\partial x\partial y} = zx^{yz-1}(1+yz\ln x),$$
$$\frac{\partial^2 u}{\partial x\partial z} = yx^{yz-1}(1+yz\ln x),$$
$$\frac{\partial^2 u}{\partial y\partial z} = (1+yz\ln x)x^{yz}\ln x.$$

(10)

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{2(1+(x_1^2+x_2^2+...+x_n^2)(2x_i^2-(x_1^2+x_2^2+...+x_n^2)))}{(1-(x_1^2+x_2^2+...+x_n^2)^2)^{\frac{3}{2}}},$$
$$\frac{\partial^2 z}{\partial x_i\partial x_j} = \frac{4x_ix_j(x_1^2+x_2^2+...+x_n^2)}{(1-(x_1^2+x_2^2+...+x_n^2)^2)^{\frac{3}{2}}}.$$

□

9.9.3 设 $u=e^{a\theta}\cos(a\ln r)$(a 为常数). 求证:

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}=0.$$

解.

$$\begin{aligned}\frac{\partial u}{\partial r}&=-e^{a\theta}\sin(a\ln r)\frac{a}{r},\\ \frac{\partial^2 u}{\partial r^2}&=e^{a\theta}(\frac{a}{r^2}\sin(a\ln r)-\frac{a^2}{r^2}\cos(a\ln r)),\\ \frac{\partial^2 u}{\partial \theta^2}&=a^2e^{a\theta}\cos(a\ln r),\end{aligned}$$

带入即有

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}=0.$$

□

9.9.4 设 u 是 x,y,z 的函数，令

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2},$$

我们称

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2},$$

为 Laplace 算子.

(1) 设 $p=\sqrt{x^2+y^2+z^2}$. 证明:

$$\Delta p=\frac{2}{p},\qquad \Delta\ln p=\frac{1}{p^2},\qquad \Delta(\frac{1}{p})=0,$$

其中 $p>0$.

(2) 设 $u=f(p)$. 求 Δu.

解. (1)

$$\begin{aligned}\frac{\partial^2 p}{\partial x^2}&=\frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}},\\ \frac{\partial^2 p}{\partial y^2}&=\frac{x^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}},\\ \frac{\partial^2 p}{\partial z^2}&=\frac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}},\end{aligned}$$

故得

$$\Delta p=\frac{2}{p},$$

同理有

$$\begin{aligned}\frac{\partial^2\ln p}{\partial x^2}&=\frac{1}{p^2}\frac{y^2+z^2-x^2}{x^2+y^2+z^2},\\ \frac{\partial^2\ln p}{\partial y^2}&=\frac{1}{p^2}\frac{x^2+z^2-y^2}{x^2+y^2+z^2},\\ \frac{\partial^2\ln p}{\partial z^2}&=\frac{1}{p^2}\frac{x^2+y^2-z^2}{x^2+y^2+z^2},\end{aligned}$$

故得

$$\Delta \ln p = \frac{1}{p^2},$$

同理有

$$\frac{\partial^2 \frac{1}{p}}{\partial x^2} = \frac{1}{p^3}\frac{-y^2 - z^2 + 2x^2}{x^2 + y^2 + z^2},$$
$$\frac{\partial^2 \frac{1}{p}}{\partial y^2} = \frac{1}{p^3}\frac{-x^2 - z^2 + 2y^2}{x^2 + y^2 + z^2},$$
$$\frac{\partial^2 \frac{1}{p}}{\partial z^2} = \frac{1}{p^3}\frac{-x^2 - y^2 + 2z^2}{x^2 + y^2 + z^2},$$

故得

$$\Delta \frac{1}{p} = 0,$$

(2)

$$\frac{\partial u}{\partial x} = f'(p)\frac{\partial p}{\partial x},$$
$$\frac{\partial^2 u}{\partial x^2} = f''(p)(\frac{\partial p}{\partial x})^2 + f'(p)\frac{\partial^2 p}{\partial x^2},$$

代入得

$$\begin{aligned}\Delta u &= f''(p)||\nabla p||^2 + f'(p)\Delta p \\ &= f''(p) + \frac{2}{p}f'(p).\end{aligned}$$

□

9.9.6 解下列方程，其 u 是 x,y,z 的函数:

(1) $\frac{\partial^2 u}{\partial x^2} = 0$;

(2) $\frac{\partial^2 u}{\partial x \partial y} = 0$;

(3) $\frac{\partial^3 u}{\partial x \partial y \partial z} = 0$.

解. (1) 由 $\frac{\partial^2 u}{\partial x^2} = 0$, 两边对 x 积分，得到

$$\frac{\partial u}{\partial x} = f(y, z),$$

两边在关于 x 积分, 得到

$$u(x, y, z) = f(y, z)x + g(y, z),$$

注：上面对于 x 计算过程中把 y,z 都试为常数。

(2) 由 $\frac{\partial^2 u}{\partial x \partial y} = 0$, 两边对 y 积分，得到

$$\frac{\partial u}{\partial x} = f(x, z),$$

两边在关于 x 积分, 得到

$$u(x, y, z) = \int f(x, z)dy = g(x, z) + h(y, z),$$

其中 g(x,z) 是关于 x 的可导函数.

(3) 由 $\frac{\partial^3 u}{\partial x\partial y\partial z}=0$, 两边对 z 积分，得到

$$\frac{\partial^2 u}{\partial x\partial y}=f(x,y),$$

两边在关于 y 积分, 得到

$$\frac{\partial u}{\partial x}=\int f(x,y)dy=g(x,y)+h(x,z),$$

两边在关于 z 积分, 得到

$$u=\int g(y,z)+h(x,z)dz=p(y,z)+q(x,z)+r(x,y),$$

其中 p(y,z) 是关于 y,z 的可导函数,q(x,z) 是关于 x,z 的可导函数. □

9.9.7 求解偏微分方程

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z.$$

解. 做换元

$$\begin{cases} x=p \\ y=\frac{p}{q}, \end{cases}$$

记换元之后的函数

$$\tilde{z}(p,q)=z(x(p,q),y(p,q)),$$

则由链式法则得到

$$\frac{\partial\tilde{z}}{\partial p}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial p}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial p},$$
$$\frac{\partial\tilde{z}}{\partial q}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial q}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial q},$$

带入即有

$$\frac{\partial\tilde{z}}{\partial p}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\frac{1}{q},$$
$$\frac{\partial\tilde{z}}{\partial q}=-\frac{\partial z}{\partial y}\frac{p}{q^2},$$

反解得到

$$\frac{\partial z}{\partial x}=\frac{\partial\tilde{z}}{\partial p}+\frac{\partial\tilde{z}}{\partial q}\frac{q}{p},$$
$$\frac{\partial z}{\partial y}=-\frac{\partial\tilde{z}}{\partial q}\frac{q^2}{p},$$

带入原方程得到

$$p\frac{\partial\tilde{z}}{\partial p}=\tilde{z},$$

解得

$$\tilde{z}(p,q)=pf(q),$$

带入即有

$$z(x,y)=xf(\frac{x}{y}).$$

□

9.9.8. 设 a,b,c 满足 $b^2-ac>0$,λ_1,λ_2 是二次方程 $cx^2+2bx+a=0$ 的两个根. 试通过引进新变量

$$\xi = x+\lambda_1 y, \qquad \eta = x+\lambda_2 y,$$

解二阶偏微分方程

$$a\frac{\partial^2 u}{\partial x^2}+2b\frac{\partial^2 u}{\partial x\partial y}+c\frac{\partial^2 u}{\partial y^2}=0.$$

解. 做换元

$$\begin{cases}\xi = x+\lambda_1 y,\\ \eta = x+\lambda_2 y,\end{cases}$$

记换元之后的函数

$$\tilde{u}(\xi,\eta)=u(x(\xi,\eta),y(\xi,\eta)),$$

则有

$$u(x,y)=\tilde{u}(\xi(x,y),\eta(x,y)),$$

则由链式法则得到

$$\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial \tilde{u}}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial \tilde{u}}{\partial \eta}\frac{\partial \eta}{\partial x},\\ \frac{\partial u}{\partial y}&=\frac{\partial \tilde{u}}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial \tilde{u}}{\partial \eta}\frac{\partial \eta}{\partial y},\end{aligned}$$

带入即有

$$\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial \tilde{u}}{\partial \xi}(\xi(x,y),\eta(x,y))+\frac{\partial \tilde{u}}{\partial \eta}(\xi(x,y),\eta(x,y)),\\ \frac{\partial u}{\partial y}&=\lambda_1\frac{\partial \tilde{u}}{\partial \xi}(\xi(x,y),\eta(x,y))+\lambda_2\frac{\partial \tilde{u}}{\partial \eta}(\xi(x,y),\eta(x,y)),\end{aligned}$$

再由链式法则求导得到

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=\frac{\partial^2 \tilde{u}}{\partial \xi^2}+2\frac{\partial^2 \tilde{u}}{\partial \xi\partial \eta}+\frac{\partial^2 \tilde{u}}{\partial \eta^2},\\ \frac{\partial^2 u}{\partial x\partial y}&=\lambda_1\frac{\partial^2 \tilde{u}}{\partial \xi^2}+(\lambda_1+\lambda_2)\frac{\partial^2 \tilde{u}}{\partial \xi\partial \eta}+\lambda_2\frac{\partial^2 \tilde{u}}{\partial \eta^2},\\ \frac{\partial^2 u}{\partial y^2}&=\lambda_1^2\frac{\partial^2 \tilde{u}}{\partial \xi^2}+2\lambda_1\lambda_2\frac{\partial^2 \tilde{u}}{\partial \xi\partial \eta}+\lambda_2^2\frac{\partial^2 \tilde{u}}{\partial \eta^2},\end{aligned}$$

带入原方程得到

$$\frac{\partial^2 \tilde{u}}{\partial \xi\partial \eta}=0,$$

解得

$$\tilde{u}(\xi,\eta)=f(\xi)+g(\eta),$$

带入即有

$$u(x,y)=f(x+\lambda_1 y)+g(x+\lambda_2 y).$$

□

7 Week 7

9.10.1 将下列多项式在指定点处展开为 Taylor 多项式 (写出前三项);
(1)$2x^2-xy-y^2-6x-3y+5$, 在点 (1,-2) 处;
(2)$x^3+y^3+z^3-3xyz$, 在点 (1,1,1) 处.

解. (1)$f(x,y)=2(x-1)^2-(x-1)(y+2)-(y+2)^2+5$
(2)$g(x,y,z)=3(x-1)^2+3(y-1)^2+3(z-1)^2-3(x-1)(y-1)-3(x-1)(z-1)-3(y-1)(z-1)$ □

9.10.2 考察二次多项式

$$f(x,y,z)=\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} A & D & F \\ D & B & E \\ F & E & C \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

试将 $f(x+\Delta x, y+\delta y, z+\delta z)$ 按 Δx,Δy,Δz 的正整数幂展开.

解. 带入有 f 为一个二次型

$$f(x,y,z)=Ax^2+By^2+Cz^2+2Dxy+2Eyz+2Fxz$$

, 则可以计算其各阶偏导数在 (x,y,z) 处的值，带入 Taylor 公式即有

$$\begin{aligned}
&f(x+\Delta x, y+\Delta y, z+\Delta z)\\
&=f(x,y,z)+Jf(x,y,z)\begin{pmatrix} x \\ y \\ z \end{pmatrix}+\frac{1}{2}(\Delta x,\Delta y,\Delta z)Hf(x,y,z)\begin{pmatrix} x \\ y \\ z \end{pmatrix}\\
&=f(x,y,z)+2\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} A & D & F \\ D & B & E \\ F & E & C \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}\\
&+\begin{pmatrix} \Delta x & \Delta y & \Delta z \end{pmatrix}\begin{pmatrix} A & D & F \\ D & B & E \\ F & E & C \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}
\end{aligned}$$

□

9.10.3 将 x^y 在点 (1,1) 处作 Taylor 展开, 写到二次项.

解.

$$f(x,y)=1+(x-1)+(x-1)(y-1)+o(||h||^2)$$

□

9.10.4 证明：当 $|x|$ 和 $|y|$ 充分小时，有近似式

$$\frac{\cos x}{\cos y}=1-\frac{1}{2}(x^2-y^2)+o(x^2+y^2).$$

解. 我们记 $f(x,y)=\frac{\cos x}{\cos y}$, 计算各界偏导数有:

$$\begin{aligned}\frac{\partial f}{\partial x}&=-\frac{\sin x}{\cos y},\\ \frac{\partial f}{\partial y}&=-\frac{\cos x\sin y}{\cos^2 y},\\ \frac{\partial^2 f}{\partial x^2}&=-\frac{\cos x}{\cos y},\\ \frac{\partial^2 f}{\partial x\partial y}&=\frac{\sin x\sin y}{\cos^2 y},\\ \frac{\partial^2 f}{\partial y^2}&=\frac{cosx(1+\sin^2 y)}{\cos^3 y},\end{aligned}$$

在 (0,0) 处取值带入 Taylor 公式得到

$$\frac{\cos x}{\cos y}=1-\frac{1}{2}(x^2-y^2)+o(x^2+y^2).$$

□

9.11.1 求下列函数的极值:

(2) $f(x,y)=x^2-3x^2y+y^3$;

(4) $f(x,y)=x^3+y^3-3xy$.

解. (2)

$$\begin{aligned}\frac{\partial f}{\partial x}&=2x(1-3y),\\ \frac{\partial f}{\partial y}&=3(y^2-x^2),\\ \frac{\partial^2 f}{\partial x^2}&=2-6y,\\ \frac{\partial^2 f}{\partial x\partial y}&=-6x,\\ \frac{\partial^2 f}{\partial y^2}&=6y,\end{aligned}$$

解驻点方程

$$\begin{cases}\frac{\partial f}{\partial x}=0,\\ \frac{\partial f}{\partial y}=0,\end{cases}$$

得到三个点 $(0,0),(\frac{1}{3},\frac{1}{3}),(-\frac{1}{3},\frac{1}{3})$ 为驻点.

带入 Hessen 矩阵发现 $(\frac{1}{3},\frac{1}{3}),(-\frac{1}{3},\frac{1}{3})$ 处的 Hessen 阵都为不定阵，故不为极值点.

考虑 (0,0) 点处,$\forall\epsilon>0$,$f(0,\epsilon)=\epsilon^3>0$, 但是 $f(0,-\epsilon)=-\epsilon^3<0$, 故 (0,0) 点处也不为极值点.

综上 f 没有极值点.

(4)

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 3x^2-3y,\\
\frac{\partial f}{\partial y} &= 3y^2-3x,\\
\frac{\partial^2 f}{\partial x^2} &= 6x,\\
\frac{\partial^2 f}{\partial x\partial y} &= -3,\\
\frac{\partial^2 f}{\partial y^2} &= 6y,
\end{aligned}$$

解驻点方程

$$\begin{cases}\frac{\partial f}{\partial x}=0,\\ \frac{\partial f}{\partial y}=0,\end{cases}$$

得到两个点 $(0,0),(1,1)$ 为驻点.
带入 Hessen 矩阵发现 $(0,0)$ 处的 Hessen 阵为不定阵，故不为极值点. 而 $(1,1)$ 处的矩阵正定, 为极小值点. □

9.11.2 求函数 $f(x,y)=xy\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}(a>0,b>0)$ 的极值.

解.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= y\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}-\frac{x^2y}{a^2\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}},\\
\frac{\partial f}{\partial y} &= x\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}-\frac{xy^2}{b^2\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}},\\
\frac{\partial^2 f}{\partial x^2} &= -\frac{3xy}{a^2\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}-\frac{x^3y}{a^4\sqrt{(1-\frac{x^2}{a^2}-\frac{y^2}{b^2})^3}},\\
\frac{\partial^2 f}{\partial x\partial y} &= \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}}{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}-\frac{x^2y^2}{a^2b^2\sqrt{(1-\frac{x^2}{a^2}-\frac{y^2}{b^2})^3}},\\
\frac{\partial^2 f}{\partial y^2} &= -\frac{3xy}{b^2\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}-\frac{xy^3}{b^4\sqrt{(1-\frac{x^2}{a^2}-\frac{y^2}{b^2})^3}},,
\end{aligned}$$

解驻点方程

$$\begin{cases}\frac{\partial f}{\partial x}=0,\\ \frac{\partial f}{\partial y}=0,\end{cases}$$

得到四个点 $(\frac{\sqrt{3}}{3}a,\frac{\sqrt{3}}{3}b),(\frac{\sqrt{3}}{3}a,-\frac{\sqrt{3}}{3}b),(-\frac{\sqrt{3}}{3}a,\frac{\sqrt{3}}{3}b),(-\frac{\sqrt{3}}{3}a,-\frac{\sqrt{3}}{3}b)$ 为驻点.
带入 Hessen 矩阵发现 $(\frac{\sqrt{3}}{3}a,\frac{\sqrt{3}}{3}b),(-\frac{\sqrt{3}}{3}a,-\frac{\sqrt{3}}{3}b)$ 处的 Hessen 阵

$$\begin{pmatrix} -\frac{4\sqrt{3}}{3}\frac{b}{a} & -\frac{2\sqrt{3}}{3}\\ -\frac{2\sqrt{3}}{3} & -\frac{4\sqrt{3}}{3}\frac{a}{b}\end{pmatrix}$$

为严格负定阵，故为极大值点.
$(\frac{\sqrt{3}}{3}a, -\frac{\sqrt{3}}{3}b), (-\frac{\sqrt{3}}{3}a, \frac{\sqrt{3}}{3}b)$ 处的 Hessen 阵

$$\begin{pmatrix} \frac{4\sqrt{3}}{3}\frac{b}{a} & -\frac{2\sqrt{3}}{3} \\ -\frac{2\sqrt{3}}{3} & \frac{4\sqrt{3}}{3}\frac{a}{b} \end{pmatrix}$$

为严格正定阵，故为极小值点.
故得到极大值点为 $\frac{\sqrt{3}ab}{9}$, 极小值点为 $-\frac{\sqrt{3}ab}{9}$. □

9.11.3 求函数

$$f(x,y) = \sin x + \cos y + \cos(x-y)$$

在正方形 $[0, \pi/2]^2$ 上的极值.

解.

$$\frac{\partial f}{\partial x} = \cos x - \sin(x-y),$$
$$\frac{\partial f}{\partial y} = -\sin y + \sin(x-y),$$
$$\frac{\partial^2 f}{\partial x^2} = -\sin x - \cos(x-y),$$
$$\frac{\partial^2 f}{\partial x \partial y} = \cos(x-y),$$
$$\frac{\partial^2 f}{\partial y^2} = -\cos y - \cos(x-y),$$

解驻点方程

$$\begin{cases} \frac{\partial f}{\partial x} = 0, \\ \frac{\partial f}{\partial y} = 0, \end{cases}$$

得到三个点 $(\frac{\pi}{3}, \frac{\pi}{6})$ 为驻点.
带入 Hessen 矩阵发现 $(\frac{\pi}{3}, \frac{\pi}{6})$ 处的 Hessen 阵

$$Hf(\frac{\pi}{3}, \frac{\pi}{6}) = \begin{pmatrix} -\sqrt{3} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\sqrt{3} \end{pmatrix}$$

为严格负定阵，故为极大值点. 则极大值为 $\frac{3\sqrt{3}}{2}$. □

9.11.4 设 $f(x,y) = 3x^4 - 4x^2y + y^2$. 证明限制在每一条过原点的直线上，原点时 f 的极值点，但是函数 f 在原点处不取极小值.

解. 若限制在直线 $y = kx$ $(k \neq 0)$ 上得到

$$g(x) = f(x, kx) = (x-k)(3x-k)x^2$$

对其求导有

$$g'(0) = 0 \qquad g''(0) = 2k^2 > 0$$

故 0 是 g 的极小值点.
若限制在 y 轴上，即 $x = 0$. 则 f 化为 y^2 也成立.
但在 $\mathbb{R}^2$ 上,$f(x,y) = (y-x^2)(y-3x^2)$,$f(0,0) = 0$, 在我们取 $y < 0$ 时,$f(x,y) > 0$, 但是取 $y = x^2$, 则有 $f(x, x^2) = -x^4 < 0$, 故原点不为 f 的极值点. □

9.11.5 设二元函数 F 在 $\mathbb{R}^2$ 上的连续可微. 已知曲线 $F(x,y)=0$ 呈 "8" 字形. 问方程组

$$\begin{cases} \frac{\partial F}{\partial x}(x,y)=0, \\ \frac{\partial F}{\partial y}(x,y)=0, \end{cases}$$

在 $\mathbb{R}^2$ 中至少有几组解?

解. 设两个圆分别为 Γ_1 和 Γ_2, 则由 Γ_1 为紧集. 则 F 在上面可以取到极值，由 $F|_{\partial\Gamma_1}=0$, 则在内部必有一个极值点. 同理在 Γ_2 的内部也有一个极值点. 而极值点一定为驻点。
设 Γ_1 和 Γ_2 相交于 p 点处. 由于 $F|_{\partial\Gamma_1}=0$，可以得到 F 沿 Γ_1 的两个方向的方向导数都为 0，又由于这两个方向线性无关. 则在 p 点处的两个偏导数也为 0，故 p 点也为驻点。
综上一共有三组解满足方程. □

8 Week 8

9.12.1 求条件极值。
(3) $u=x-2y+2z, x^2+y^2+z^2=1$.
(4) $u=3x^2+3y^2+z^2, x+y+z=1$.

解. (3) $F(x,y,z)=x-2y+2z-\lambda(x^2+y^2+z^2-1)$

$$\begin{cases} \dfrac{\partial F}{\partial x}=1-2\lambda x=0 \\ \dfrac{\partial F}{\partial y}=-2-2\lambda y=0 \\ \dfrac{\partial F}{\partial z}=2-2\lambda z=0 \end{cases} \Rightarrow \begin{cases} x=\dfrac{1}{2\lambda} \\ y=-\dfrac{1}{\lambda} \\ z=\dfrac{1}{\lambda} \end{cases}$$

代入 $x^2+y^2+z^2=1$ 得 $\dfrac{1}{4\lambda^2}+\dfrac{1}{\lambda^2}+\dfrac{1}{\lambda^2}=\dfrac{9}{4\lambda^2}=1$, 得 $\lambda=\pm\dfrac{3}{2}$,$(x,y,z)=\left(\dfrac{1}{3},-\dfrac{2}{3},\dfrac{2}{3}\right)$ 或 $\left(-\dfrac{1}{3},\dfrac{2}{3},-\dfrac{2}{3}\right)$.

$$\mathbf{H}u=\begin{pmatrix} -2\lambda & & \\ & -2\lambda & \\ & & -2\lambda \end{pmatrix}$$

$\lambda<0$ 时严格正定，$\lambda>0$ 时严格负定，故 $\left(\dfrac{1}{3},-\dfrac{2}{3},\dfrac{2}{3}\right)$ 是极大值点，极大值为 3；$\left(-\dfrac{1}{3},\dfrac{2}{3},-\dfrac{2}{3}\right)$ 是极小值点，极小值为 -3.

(4) $F(x,y,z)=3x^2+3y^2+z^2-\lambda(x+y+z-1)$

$$\begin{cases} \dfrac{\partial F}{\partial x}=6x-\lambda=0 \\ \dfrac{\partial F}{\partial y}=6y-\lambda=0 \\ \dfrac{\partial F}{\partial z}=2z-\lambda=0 \end{cases} \Rightarrow \begin{cases} x=\dfrac{\lambda}{6} \\ y=\dfrac{\lambda}{6} \\ z=\dfrac{\lambda}{2} \end{cases}$$

代入 $x+y+z=1$ 得 $\dfrac{5}{6}\lambda=1$, 得 $\lambda=\dfrac{6}{5}$, $(x,y,z)=\left(\dfrac{1}{5},\dfrac{1}{5},\dfrac{3}{5}\right)$.

$$\mathbf{H}u=\begin{pmatrix} 6 & & \\ & 6 & \\ & & 2 \end{pmatrix}$$

严格正定，故 $\left(\frac{1}{5},\frac{1}{5},\frac{3}{5}\right)$ 是极小值点，极小值为 $\frac{3}{5}$. □

9.12.2 计算

(1) 原点到 $\begin{cases} 2x+2y+z+9=0 \\ 2x-y-2z-18=0 \end{cases}$ 的距离。

(2) 原点到 $x+2y+3z+4=0$ 的距离。

解. (1) $d^2=x^2+y^2+z^2, F(x,y,z)=x^2+y^2+z^2-\lambda_1(2x+2y+z+9)-\lambda_2(2x-y-2z-18)$

$$\begin{cases} \frac{\partial F}{\partial x}=2x-2\lambda_1-2\lambda_2=0 \\ \frac{\partial F}{\partial y}=2y-2\lambda_1+\lambda_2=0 \\ \frac{\partial F}{\partial z}=2z-\lambda_1+2\lambda_2=0 \end{cases} \Rightarrow \begin{cases} x=\lambda_1+\lambda_2 \\ y=\lambda_1-\frac{\lambda_2}{2} \\ z=\frac{\lambda_1}{2}-\lambda_2 \end{cases}$$

代入 $\begin{cases} 2x+2y+z+9=0 \\ 2x-y-2z-18=0 \end{cases}$ 得 $\begin{cases} \frac{9}{2}\lambda_1+9=0 \\ \frac{9}{2}\lambda_2-18=0 \end{cases} \Rightarrow \begin{cases} \lambda_1=-2 \\ \lambda_2=4 \end{cases}$,

于是 $\begin{cases} x=2 \\ y=-4 \\ z=-5 \end{cases}$. 由于 $\mathbf{H}f=\begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix}$ 恒为严格正定的，该点是 d^2 的极小值点，于是距离为 $d_{\min}=\sqrt{2^2+4^2+5^2}=3\sqrt{5}$.

(2) $F(x,y,z)=x^2+y^2+z^2-\lambda(x+2y+3z+4)$

$$\begin{cases} \frac{\partial F}{\partial x}=2x-\lambda=0 \\ \frac{\partial F}{\partial y}=2y-2\lambda=0 \\ \frac{\partial F}{\partial z}=2z-3\lambda=0 \end{cases} \Rightarrow \begin{cases} x=\frac{\lambda}{2} \\ y=\lambda \\ z=\frac{3\lambda}{2} \end{cases}$$

代入 $x+2y+3z+4=0$ 得 $\lambda=-\frac{4}{7}, (x,y,z)=-\frac{2}{7}(1,2,3), \mathbf{H}f$ 同上，恒为正定，故该点是极小值点，得到距离为

$$d_{\min}=\frac{2}{7}\sqrt{1+2^2+3^2}=\frac{2}{7}\sqrt{14}.$$

□

9.12.4 设 $a>0$, 求 $\begin{cases} x^2+y^2=2az \\ x^2+y^2+xy=a^2 \end{cases}$ 上的点到 Oxy 平面的最小距离和最大距离。

解. $F(x,y,z)=z^2-\lambda_1(x^2+y^2-2az)-\lambda_2(x^2+y^2+xy-a^2)$

$$\begin{cases} \frac{\partial F}{\partial x}=-2\lambda_1x-2\lambda_2x-\lambda_2y=0 \\ \frac{\partial F}{\partial y}=-2\lambda_1y-2\lambda_2y-\lambda_2x=0 \\ \frac{\partial F}{\partial z}=2z+2a\lambda_1=0 \end{cases}$$

前两个方程整理得

$$\begin{cases} 2(\lambda_1+\lambda_2)x+\lambda_2 y=0 \\ \lambda_2 x+2(\lambda_1+\lambda_2)y=0 \end{cases}$$

由于 (x,y) 要满足 $x^2+y^2+xy=a^2>0$, 故 $(x,y)\neq(0,0)$, 因而上述方程组有非零解，即系数矩阵得行列式 $=0$.

$$4(\lambda_1+\lambda_2)^2-\lambda_2^2=0,$$

得

$$2\lambda_1+3\lambda_2=0 \text{ or } 2\lambda_1+\lambda_2=0.$$

若 $2\lambda_1+3\lambda_2=0$, 则 $x=y$,

$$\begin{cases} x^2+y^2=2az \\ 3x^2=a^2 \end{cases}$$

得 $z=\dfrac{a}{3}$ 为极小值。

若 $2\lambda_1+\lambda_2=0$, 则 $x=-y$,

$$\begin{cases} 2x^2=2az \\ x^2=a^2 \end{cases}$$

得 $z=a$ 为极大值。 □

9.12.6 设 $a_i\geq 0, i=1,2,\cdots,n, p>1$, 证明:

$$\frac{a_1+\cdots+a_n}{n}\leq\Big(\frac{a_1^p+\cdots+a_n^p}{n}\Big)^{\frac{1}{p}}.$$

解. 设 $a_1+\cdots+a_n=c, f(a_1,\cdots,a_n)=a_1^p+\cdots+a_n^p-\lambda(a_1+\cdots+a_n-c)$,

$$\frac{\partial f}{\partial a_i}=pa_i^{p-1}-\lambda=0\Rightarrow a_i^{p-1}=\frac{\lambda}{p}\Rightarrow a_i=\Big(\frac{\lambda}{p}\Big)^{\frac{1}{p-1}},\forall i\Rightarrow c=n\Big(\frac{\lambda}{p}\Big)^{\frac{1}{p-1}}\Rightarrow a_i=\frac{c}{n},\forall i$$

若 $c=0$ 则不等式显然成立，否则 $c>0$,f 的 Hesse 阵严格正定，故有

$$\frac{a_1+\cdots+a_n}{n}=\frac{c}{n}\leq\Big(\frac{a_1^p+\cdots+a_n^p}{n}\Big)^{\frac{1}{p}}.$$

□

9.12.7 证明：设 $a_i\geq 0, x_i\geq 0(i=1,2,\cdots,n), p>1, \dfrac{1}{p}+\dfrac{1}{q}=1$, 则

$$\sum_{i=1}^n a_i x_i\leq\Big(\sum_{i=1}^n a_i^p\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^n x_i^q\Big)^{\frac{1}{q}}$$

解. 设 $\sum\limits_{i=1}^n a_i x_i=c, f(x_1,\cdots,x_n)=x_1^q+\cdots+x_n^q-\lambda\big(\sum\limits_{i=1}^n a_i x_i-c\big)$.

$$\frac{\partial f}{\partial x_i}=qx_i^{q-1}-\lambda a_i=0\Rightarrow x_i^{q-1}=\frac{\lambda a_i}{q},$$

代入 $\sum\limits_{i=1}^n a_i x_i = c$ 得

$$\frac{\lambda}{q} = \left(\frac{c}{\sum\limits_{i=1}^n a_i^p}\right)^{q-1}$$

故

$$\sum_{i=1}^n x_i^q \geq \sum_{i=1}^n \left(\frac{\lambda a_i}{q}\right)^{\frac{q}{q-1}} = \left(\frac{\lambda}{q}\right)^{\frac{q}{q-1}} \sum_{i=1}^n a_i^{\frac{q}{q-1}} = Big(\frac{\lambda}{q}\Big)^{\frac{q}{q-1}} \sum_{i=1}^n a_i^p = \left(\frac{\sum\limits_{i=1}^n a_i x_i}{\sum\limits_{i=1}^n a_i^p}\right)^q \sum_{i=1}^n a_i^p$$

整理后得

$$\sum_{i=1}^n a_i x_i \leq \Big(\sum_{i=1}^n a_i^p\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^n x_i^q\Big)^{\frac{1}{q}}.$$

□

9 Week 9

10.1.1 一元函数 f, g 在区间 $[0,1]$ 可积，求证 $f(x)g(y)$ 在 $I = [0,1]^2$ 上可积，且

$$\iint_I f(x)g(y)\mathrm{d}x\mathrm{d}y = \int_0^1 f(x)\mathrm{d}x \int_0^1 g(y)\mathrm{d}y$$

解. 记在区域 I 上的分割为 π,x 方向的分割为 π_x,y 方向的分割为 π_y.

f, g 可积，故 f, g 均有界且

$$\lim_{||\pi_x||\to 0} \sum_{i=1}^m \omega_i(f) \triangle x_i = 0,$$
$$\lim_{||\pi_y||\to 0} \sum_{j=1}^m \omega_j(g) \triangle y_j = 0.$$

对任意的 x_1, x_2, y_1, y_2,

$$\begin{aligned}|f(x_1)g(y_1) - f(x_2)g(y_2)| &= |f(x_1)g(y_1) - f(x_1)g(y_2) + f(x_1)g(y_2) - f(x_2)g(y_2)| \\ &\leq |f(x_1)| \cdot |g(y_1) - g(y_2)| + |g(y_2)| \cdot |f(x_1) - f(x_2)|\end{aligned}$$

同时分割的每个小区域内取 sup 得 $\omega_{ij}(fg) \leq M\big(\omega_i(f) + \omega_j(g)\big)$, 其中 M 为 $|f|$ 和 $|g|$ 的一个共同上界。

$$\begin{aligned}\sum_{i=1}^m \sum_{j=1}^n \omega_{ij}(fg)\sigma(I_{ij}) &= \sum_{i=1}^m \sum_{j=1}^n \omega_{ij}(fg) \triangle x_i \triangle y_j \\ &\leq M\Big(\sum_{i=1}^m \omega_i(f) \triangle x_i + \sum_{j=1}^n \omega_j(g) \triangle y_j\Big)\end{aligned}$$

$||\pi|| \to 0$ 时，$||\pi_x||, ||\pi_y|| \to 0$, 故

$$\lim_{||\pi||\to 0} \sum_{i=1}^m \sum_{j=1}^n \omega_{ij}(fg)\sigma(I_{ij}) = 0.$$

从而可积。 □

10.1.2 计算 $\iint_{[0,1]^2} e^{x+y}\mathrm{d}x\mathrm{d}y$

解.

$$\iint_{[0,1]^2} e^{x+y}\mathrm{d}x\mathrm{d}y = \int_0^1 e^x\mathrm{d}x\int_0^1 e^y\mathrm{d}y = (e-1)^2.$$

□

10.1.3 $a>0, I-[-a,a]^2$, 求证:$\iint_I \sin(x+y)\mathrm{d}x\mathrm{d}y = 0$.

解. 做变换 $t=-x, s=-y$,

$$\begin{aligned}\iint_I \sin(x+y)\mathrm{d}x\mathrm{d}y &= \int_{-a}^{a}\int_{-a}^{a}\sin(x+y)\mathrm{d}x\mathrm{d}y\\ &= \int_{-a}^{a}\int_{-a}^{a}\sin(-s-t)\mathrm{d}t\mathrm{d}s\\ &= -\int_{-a}^{a}\int_{-a}^{a}\sin(s+t)\mathrm{d}t\mathrm{d}s\end{aligned}$$

故 $\iint_I \sin(x+y)\mathrm{d}x\mathrm{d}y = 0$. □

10.1.5 证明：闭矩形上的连续函数可积。

解. 闭矩形上的连续函数必然有界且一致连续，故 $\forall\varepsilon>0, \exists\delta>0$, 当 $||\boldsymbol{x}-\boldsymbol{y}||<\delta$ 时，$|f(\boldsymbol{x})-f(\boldsymbol{y})|<\varepsilon$. 故 $||\pi||<\varepsilon$ 时，有 $\omega_i<\varepsilon$, 故

$$\sum_i \omega_i\sigma(I_i) < \sigma(I)\varepsilon \Rightarrow \lim_{||\pi||\to 0}\sum_i \omega_i\sigma(I_i) = 0 \Rightarrow \text{ 可积.}$$

□

10.2.2 设有界集 $B\subset\mathbb{R}^2$ 且 B' 是零面积集，求证 $\overline{B}$ 也是零面积集。

解. $\overline{B} = B'\cup(\partial B\backslash B')$, 由于 B' 是零面积集，$\forall\varepsilon>0, \exists I_1,\cdots,I_m, B'\subset\bigcup\limits_{i=1}^{m} I_i, \sum\limits_{i=1}^{m}\sigma(I_i)<\varepsilon$.

$\partial B\backslash B'$ 为孤立点, 设 $\partial B\backslash B' = \{a_1,\cdots,a_n,\cdots\}$,$\forall i\in N, \exists W_{a_i}$ 为 a_i 的邻域, 满足 $\sigma(W_{a_i})<\dfrac{\varepsilon}{2^i}$, 则 W_{a_i} 是 $\partial B\backslash B'$ 的开覆盖，由于 B 有界，故 $\partial B\backslash B'$ 是有界闭集，因此上述开覆盖存在一个有限子覆盖记为 $W_1,\cdots,W_N$, 有 $\sigma(W_1)+\cdots+\sigma(W_N)<\varepsilon$.

令 $I_{m+i}=W_i, i=1,\cdots,N$, 则 $\overline{B}\subset\bigcup\limits_{i=1}^{m+N} I_i$, 且 $\sum\limits_{i=1}^{m+N}\sigma(I_i)<2\varepsilon$. 故 $\overline{B}$ 是零面积集。 □

10.2.4 闭矩形 $J\subset I$,f 在 I 上可积，求证 f 在 J 上也可积。

解. $J\subset I$, 故若 x 是 f 在 J 上的间断点，则 x 也是 f 在 I 上的间断点，即 $D_J(f)\subset D_I(f)$. f 在 I 上可积，则 f 有界且 $D_I(f)$ 零测，故 $D_J(f)$ 零测，因而 f 在 J 上可积。 □

10.2.5 I 上可积函数 $f>0$, 求证 $\int_I f\mathrm{d}\sigma>0$.

解. f 可积，故 f 有界且 $D(f)$ 零测，取 $x_0 \in I \backslash D(f)$, 则 $f(x_0) > 0$ 且 x_0 是连续点，故存在 x_0 的邻域 $U_0 \subset I$, 当 $x \in U_0$ 时，有 $f(x) > f(x_0)/2$. 则

$$\int_I f\mathrm{d}\sigma \geq \int_{U_0} f\mathrm{d}\sigma \geq \frac{1}{2}f(x_0)\sigma(U_0) > 0.$$

□

10.2.6 $I = [0,1]^2, f(x,y) = \begin{cases} \sin\dfrac{1}{xy}, & x \neq 0 \text{ 且} y \neq 0 \\ 0, & x = 0 \text{ or } y = 0 \end{cases}$ 的可积性。

解. $\lim\limits_{x\to 0}\sin\dfrac{1}{xy}$ 和 $\lim\limits_{y\to 0}\sin\dfrac{1}{xy}$ 不存在，在 $x \neq 0$ 且$y \neq 0$ 时均为连续函数，故 $D(f) = \{0\} \times [0,1] \cup [0,1] \times \{0\}$ 为 $\mathbb{R}^2$ 中的零测集，又 f 有界，故 f 在 I 上可积。 □

10.2.7 I 有界矩形，$B = \{\boldsymbol{p}_1, \cdots, \boldsymbol{p}_n, \cdots\} \subset I$, 函数 $f(\boldsymbol{p}) = \begin{cases} 0, & \boldsymbol{p} \notin B \\ \dfrac{1}{n}, & \boldsymbol{p} = \boldsymbol{p}_n \end{cases}$ 的可积性。

解. f 有界，且 $D(f) = B$ 零测，故 f 在 I 上可积。 □

10.2.8 f, g 在 I 上可积，则 fg 可积，$\dfrac{f}{g}$ 在 $g \neq 0$ 且有界时可积。

解. f, g 在 I 上可积, 则 f, g 在 I 上有界，则 fg 在 I 上有界。
若 f, g 连续，则 fg 连续，当 $g \neq 0$ 时 $\dfrac{f}{g}$ 连续，因此

$$\begin{aligned} &D(fg) \subset D(f) \cup D(g) \\ &D\left(\frac{f}{g}\right) \subset D(f) \cup D(g) \text{ if } g \neq 0 \end{aligned}$$

因此 fg 可积，$\dfrac{f}{g}$ 在 $g \neq 0$ 且有界时可积。 □

10.3.1 计算积分。
(1) $\displaystyle\iint_I \frac{x^2}{1+y^2}\mathrm{d}x\mathrm{d}y, I = [0,1]^2;$
(2) $\displaystyle\iint_I x\cos xy\mathrm{d}x\mathrm{d}y, I = [0,\frac{\pi}{2}] \times [0,1];$
(3) $\displaystyle\iint_I \sin(x+y)\mathrm{d}x\mathrm{d}y, I = [0,\pi]^2.$

解.

$$
\begin{aligned}
(1)\qquad \iint_I \frac{x^2}{1+y^2}\mathrm{d}x\mathrm{d}y &= \int_0^1 x^2\mathrm{d}x\int_0^1 \frac{1}{y^2+1}\mathrm{d}y \\
&= \frac{1}{3}\arctan y\Big|_0^1 = \frac{\pi}{12}. \\
(2)\qquad \iint_I x\cos xy\mathrm{d}x\mathrm{d}y &= \int_0^{\frac{\pi}{2}}\int_0^1 x\cos xy\mathrm{d}y\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \sin x\mathrm{d}x = 1. \\
(3)\quad \iint_I \sin(x+y)\mathrm{d}x\mathrm{d}y &= \int_0^{\pi}\int_0^{\pi} \sin(x+y)\mathrm{d}x\mathrm{d}y \\
&= \int_0^{\pi} \cos y - \cos(y+\pi)\mathrm{d}y \\
&= 2\int_0^{\pi}\cos y\mathrm{d}y = 0.
\end{aligned}
$$

□

10.3.2 f 在 $I=[a,b]\times[c,d]$ 上有连续的二阶导，计算 $\displaystyle\iint_I \frac{\partial^2}{\partial x\partial y}f(x,y)\mathrm{d}x\mathrm{d}y$.

解.

$$
\begin{aligned}
\iint_I \frac{\partial^2}{\partial x\partial y}f(x,y)\mathrm{d}x\mathrm{d}y &= \int_c^d\int_a^b \frac{\partial}{\partial x}\frac{\partial}{\partial y}f(x,y)\mathrm{d}x\mathrm{d}y \\
&= \int_c^d \frac{\partial}{\partial y}f(b,y) - \frac{\partial}{\partial y}f(a,y)\mathrm{d}y \\
&= f(b,d)-f(b,c)-f(a,d)+f(a,c).
\end{aligned}
$$

□

10.3.3 计算 $\displaystyle\int_I f\mathrm{d}\sigma, I=[0,1]^2$.

(1) $f(x,y)=\begin{cases}1, & y\le x^2\\ 0, & y>x^2\end{cases}$; (2) $f(x,y)=\begin{cases}x+y, & x^2\le y\le 2x^2\\ 0, & \text{else}\end{cases}$.

解.

$$
(1)\quad \int_I f\mathrm{d}\sigma = \int_0^1\int_0^{x^2} 1\mathrm{d}y\mathrm{d}x = \int_0^1 x^2\mathrm{d}x = \frac{1}{3}.
$$

$$
\begin{aligned}
(2)\quad \int_I f\mathrm{d}\sigma &= \int_0^{\frac{\sqrt{2}}{2}}\int_{x^2} 2x^2x+y\mathrm{d}y\mathrm{d}x + \int_{\frac{\sqrt{2}}{2}}^1\int_{x^2} 2x^2x+y\mathrm{d}y\mathrm{d}x \\
&= \int_0^{\frac{\sqrt{2}}{2}} x^3+\frac{3}{2}x^4\mathrm{d}x + \int_{\frac{\sqrt{2}}{2}}^1 \frac{1}{2}+x-x^3-\frac{x^4}{2}\mathrm{d}x \\
&= \frac{21}{40}-\frac{\sqrt{2}}{5}.
\end{aligned}
$$

□

10 Week 10

10.3.4 利用定理 10.3.4 的 Minkowski 不等式，证明：$a_k \geq 0, b_k \geq 0, k = 1, 2, \cdots, n, p \geq 1$, 有

$$\Big(\sum_{k=1}^{n}(a_k+b_k)^p\Big)^{\frac{1}{p}} \leq \Big(\sum_{k=1}^{n} a_k^p\Big)^{\frac{1}{p}} + \Big(\sum_{k=1}^{n} b_k^p\Big)^{\frac{1}{p}}.$$

解. 令 $f(x,y) = \begin{cases} a_k, & 0 \geq y \geq 1, k-1 \geq x \geq k \\ b_k, & 1 \geq y \geq 2, k-1 \geq x \geq k \end{cases}$, 代入定理 10.3.4 中左右两式得

$$\begin{aligned}
\Big(\int_0^n \Big(\int_0^2 f(x,y)\mathrm{d}y\Big)^p\Big)^{\frac{1}{p}} &= \Big(\sum_{k=1}^{n}\int_{k-1}^{k}(a_k+b_k)^p\mathrm{d}x\Big)^{\frac{1}{p}} = \Big(\sum_{k=1}^{n}(a_k+b_k)^p\mathrm{d}x\Big)^{\frac{1}{p}} \\
\int_0^2 \Big(\int_0^n f^p(x,y)\mathrm{d}x\Big)^{\frac{1}{p}}\mathrm{d}y &= \int_0^1 \Big(\sum_{k=1}^{n}\int_{k-1}^{k} a_k^p\Big)^{\frac{1}{p}}\mathrm{d}y + \int_1^2 \Big(\sum_{k=1}^{n}\int_{k-1}^{k} b_k^p\Big)^{\frac{1}{p}}\mathrm{d}y \\
&= \Big(\sum_{k=1}^{n} a_k^p\Big)^{\frac{1}{p}} + \Big(\sum_{k=1}^{n} b_k^p\Big)^{\frac{1}{p}}.
\end{aligned}$$

故

$$\Big(\sum_{k=1}^{n}(a_k+b_k)^p\Big)^{\frac{1}{p}} \leq \Big(\sum_{k=1}^{n} a_k^p\Big)^{\frac{1}{p}} + \Big(\sum_{k=1}^{n} b_k^p\Big)^{\frac{1}{p}}.$$

□

10.3.5 f, g 都在 $[a,b]$ 上可积，用 $\iint_{[a,b]^2} \big(f(x)g(y) - f(y)g(x)\big)^2 \mathrm{d}x\mathrm{d}y \geq 0$ 证明：

$$\Big(\int_a^b f(x)g(x)\mathrm{d}x\Big)^2 \leq \int_a^b f(x)^2\mathrm{d}x \int_a^b g(x)^2\mathrm{d}x.$$

解.

$$\begin{aligned}
0 &\leq \iint_{[a,b]^2} \big(f(x)g(y) - f(y)g(x)\big)^2 \mathrm{d}x\mathrm{d}y \\
&= \iint_{[a,b]^2} \big(f(x)g(y)\big)^2 - 2f(x)f(y)g(x)g(y) + \big(f(y)g(x)\big)^2 \mathrm{d}x\mathrm{d}y \\
&= \int_a^b f(x)^2\mathrm{d}x \int_a^b g(y)^2\mathrm{d}y - 2\int_a^b f(x)g(x)\mathrm{d}x \int_a^b f(y)g(y)\mathrm{d}y + \int_a^b f(y)^2\mathrm{d}y \int_a^b g(x)^2\mathrm{d}x \\
&= 2\int_a^b f(x)^2\mathrm{d}x \int_a^b g(x)^2\mathrm{d}x - 2\Big(\int_a^b f(x)g(x)\mathrm{d}x\Big)^2
\end{aligned}$$

□

10.4.2 证明：$1.96 < \displaystyle\iint_{|x|+|y|\leq 10} \frac{\mathrm{d}x\mathrm{d}y}{100+\cos^2 x+\cos^2 y} < 2.$

解. $E := \{(x,y) : |x|+|y| \leq 10\}, \sigma(E) = 200$, 又有 $\dfrac{1}{102} \leq \dfrac{1}{100+\cos^2 x+\cos^2 y} \leq \dfrac{1}{100}$, 故

$$1.96 < \frac{200}{102} \leq \iint_{|x|+|y|\leq 10} \frac{\mathrm{d}x\mathrm{d}y}{100+\cos^2 x+\cos^2 y} \leq \frac{200}{100} = 2,$$

由于 $\dfrac{1}{100+\cos^2 x+\cos^2 y}$ 的最大值和最小值不是处处取到，因而

$$1.96 < \iint\limits_{|x|+|y|\le 10} \frac{\mathrm{d}x\mathrm{d}y}{100+\cos^2 x+\cos^2 y} < 2.$$

□

10.4.3 $B\subset\mathbb{R}^2$ 是有界集，证明 B 有面积集当且仅当对 $I^\circ\supset B$ 的矩形 I 的任何矩形分割 π，均有 $\lim\limits_{||\pi||\to 0}\sum\limits_{I_j\cap B\neq\emptyset}\sigma(I_j)=\lim\limits_{||\pi||\to 0}\sum\limits_{I_j\subset B}\sigma(I_j)$.

解. B 有面积当且仅当 ∂B 是零面积集。对任何矩形分割 π，

$$\lim_{||\pi||\to 0}\sum_{I_j\cap B\neq\emptyset}\sigma(I_j)=\lim_{||\pi||\to 0}\sum_{I_j\subset B}\sigma(I_j)+\lim_{||\pi||\to 0}\sum_{\substack{I_j\cap B\neq\emptyset\\ I_j\cap B^c\neq\emptyset}}\sigma(I_j),$$

故 $\lim\limits_{||\pi||\to 0}\sum\limits_{I_j\cap B\neq\emptyset}\sigma(I_j)=\lim\limits_{||\pi||\to 0}\sum\limits_{I_j\subset B}\sigma(I_j)$ 当且仅当 $\lim\limits_{||\pi||\to 0}\sum\limits_{\substack{I_j\cap B\neq\emptyset\\ I_j\cap B^c\neq\emptyset}}\sigma(I_j)=0$. 由定义可知该式等价于 ∂B 是零测集，又 B 是有界集，故 ∂B 是有界闭集，因而 ∂B 是零测集等价于 ∂B 是零面积集。 □

11 Week 11

10.5.1 计算下列积分：

(2) $\iint_D xy^2dxdy$, D 由 $y^2=4x$ 和 $x=1$ 围成；

(4) $\iint_D |xy|dxdy$, $D=\{(x,y):x^2+y^2\le a^2\}$；

(6) $\iint_D |\cos(x+y)|\,dxdy$, $D=[0,1]^2$；

(7) $\iint_D y^2dxdy$, D 由滚轮线

$$\begin{cases} x=a(t-\sin t), \\ y=a(1-\sin t) \end{cases} \qquad (0\le t\le 2\pi)$$

与 $y=0$ 围成;

(8) $\iint_D [x+y]dxdy$, $D=[0,2]^2$.

解. (2)

$$\begin{aligned}\iint_D xy^2dxdy &= \int_{-2}^{2}dy\int_{\frac{y^2}{4}}^{1}xy^2dxdy \\ &= \frac{1}{2}\int_{-2}^{2}y^2(1-\frac{1}{16}y^4)dxdy \\ &= \frac{32}{21}.\end{aligned}$$

(4)

$$\iint_D |xy|dxdy = 4\iint\limits_{\substack{x^2+y^2\le a^2\\ x,y\ge 0}} xydxdy$$
$$= 4\int_0^a ydy\int_0^{\sqrt{a^2-y^2}} xdx$$
$$= 2\int_0^a y(a^2-y^2)dy$$
$$= \frac{1}{2}a^4.$$

(6) 记 $E = \{(x,y)\in D|x+y\le \frac{\pi}{2}\}$ $\quad F = \{(x,y)\in D|x+y > \frac{\pi}{2}\}$,

$$\iint\limits_D |\cos(x+y)|dxdy = \iint\limits_E |\cos(x+y)|dxdy - \iint\limits_F |\cos(x+y)|dxdy$$
$$= \iint\limits_D |\cos(x+y)|dxdy - 2\iint\limits_F |\cos(x+y)|dxdy$$
$$= \int_0^1 dy\int_0^1 \cos(x+y)dx - 2\int_{\frac{\pi}{2}-1}^1 dx\int_{\frac{\pi}{x}-x}^1 cos(x+y)dy$$
$$= \cos 2 + 2\cos 1 + 3 - \pi.$$

(7)

$$\iint\limits_D y^2dxdy = \int_0^{2\pi a} dx\int_0^{y(x)} y^2dxdy$$
$$= \int_0^{2\pi a}\frac{1}{3}(y(x))^3dx$$
$$= \int_0^{2\pi}\frac{1}{3}(y(x(t)))^3\frac{dx}{dt}dt$$
$$= \int_0^{2\pi}\frac{a^4}{3}(1-\cos t)^4dt$$
$$= \frac{35\pi a^4}{12}.$$

(8)

$$\iint\limits_D [x+y]dxdy = 6.$$

□

10.5.2 改变下列累次积分的次序:

(1) $\int_0^1 dx\int_0^{x^2} f(x,y)dy$;

(3) $\int_0^1 dx\int_{x^2}^{x} f(x,y)dy$;

(5) $\int_{-1}^1 dx\int_{-\sqrt{1-x^2}}^{1-x^2} f(x,y)dy$;

(7) $\int_0^{\pi} dx\int_0^{\cos x} f(x,y)dy$.

解. (1) $\int_0^1 dy \int_{\sqrt{y}}^1 f(x,y)dx$;

(3) $\int_0^1 dy \int_y^{\sqrt{y}} f(x,y)dx$;

(5) $\int_{-1}^0 dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)dx + \int_0^1 dy \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y)dx$;

(7) $\int_0^1 dy \int_0^{\arccos y} f(x,y)dx - \int_{-1}^0 dy \int_{\arccos y}^{\pi} f(x,y)dx$.

□

10.5.3 设 f 为一元连续函数. 求证:

$$\int_0^a dx \int_0^x f(x)f(y)dy = \frac{1}{2}\Big(\int_0^a f(t)dt\Big)^2.$$

解. 记 $F(x) = \int_0^x f(t)dt$,

$$\begin{aligned}\int_0^a dx \int_0^x f(x)f(y)dxdy &= \int_0^a dx \int_0^x f(y)dxdy \\ &= \int_0^a f(x)F(x)dx \\ &= \frac{1}{2}F(a)^2 \\ &= \frac{1}{2}\Big(\int_0^a f(t)dt\Big)^2.\end{aligned}$$

□

10.5.4 设 f 为连续函数. 证明:

$$\int_0^a dx \int_0^x f(y)dy = \int_0^a (a-t)f(t)dt.$$

解.

$$\begin{aligned}\int_0^a dx \int_0^x f(y)dy &= \int_0^a f(y)dy \int_y^a dx \\ &= \int_0^a (a-y)f(y)dy.\end{aligned}$$

□

10.5.5 设 f 为连续函数. 证明:

$$\int_a^b dx \int_a^x dy \int_a^y f(x,y,z)dz = \int_a^b dz \int_z^b dy \int_y^b f(x,y,z)dx.$$

解. 交换积分顺序即得

$$\int_a^b dx \int_a^x dy \int_a^y f(x,y,z)dz = \int_a^b dz \int_z^b dy \int_y^b f(x,y,z)dx.$$

□

10.5.6 设 f 为连续函数. 证明:

$$\int_0^a dx\int_0^x dy\int_0^y f(x)f(y)f(z)dz=\frac{1}{3!}\Big(\int_0^a f(t)dt\Big)^3.$$

解. 记 $F(x)=\int_0^a f(t)dt$,

$$\begin{aligned}\int_0^a dx\int_0^x dy\int_0^y f(x)f(y)f(z)dz &= \int_0^a f(x)dx\int_0^x F(y)f(y)dy\\ &= \int_0^a \frac{1}{2}f(x)\big(F(x)^2-F(0)^2\big)dx\\ &= \frac{1}{2}\int_0^a F(x)^2f(x)dx\\ &= \frac{1}{3!}\big(F(a)^3-F(0)^3\big)\\ &= \frac{1}{3!}\Big(\int_0^a f(t)dt\Big)^3.\end{aligned}$$

□

10.5.7 设 f 为连续函数. 证明:

$$\int_0^a dx\int_0^x dy\int_0^y f(z)dz=\frac{1}{2}\int_0^a (a-t)^2f(t)dt.$$

解.

$$\begin{aligned}\int_0^a dx\int_0^x dy\int_0^y f(z)dz &= \int_0^a dz\int_z^a dy\int_y^a f(x)dx\\ &= \int_0^a f(z)dz\int_z^a (a-y)dy\\ &= \frac{1}{2}\int_0^a (a-z)^2f(z)dz.\end{aligned}$$

□

10.5.8 设 f 为连续函数. 求极限

$$\lim_{r\to 0}\frac{1}{\pi r^2}\iint\limits_{x^2+y^2\le r^2} f(x,y)dxdy.$$

解. 由积分中值定理得

$$\begin{aligned}\lim_{r\to 0}\frac{1}{\pi r^2}\iint\limits_{x^2+y^2\le r^2} f(x,y)dxdy &= \lim_{r\to 0}\frac{1}{\pi r^2}f(\xi,\eta)\pi r^2\\ &= \lim_{r\to 0}f(\xi,\eta)\\ &= f(0,0).\end{aligned}$$

□

10.6.1 计算下列二重积分:

(1) $\iint_D (x-y)^2\sin(x+y)dxdy$, 其中 D 是由四点 $(\pi,0),(2\pi,\pi),(\pi,2\pi),(0,\pi)$ 顺次连成的正方形.

(2) $\iint_D (x^2+y^2)dxdy$, 其中 D 是由曲线 $x^2-y^2=1$,$x^2-y^2=2$,$xy=1$ 和 $xy=2$ 围成的图像在第一象限的那一部分.

解. (1) 令

$$\begin{cases} u = x - y \\ v = x + y, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2},$$

对应的积分区域化为

$$F = [-\pi, \pi] \times [\pi, 3\pi],$$

带入由换元公式得到

$$\begin{aligned} \iint\limits_D (x-y)^2 \sin(x+y) dxdy &= \iint\limits_F u^2 \sin v |\frac{1}{2}| dudv \\ &= \frac{1}{2} \int_{-\pi}^{\pi} u^2 du \int_{\pi}^{3\pi} \sin v dv \\ &= 0. \end{aligned}$$

(2) 令

$$\begin{cases} u = x^2 - y^2 \\ v = xy, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2+y^2) \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2(x^2+y^2)},$$

对应的积分区域化为

$$F = [1, 2] \times [1, 2],$$

带入由换元公式得到

$$\begin{aligned} \iint\limits_D x^2 + y^2 dxdy &= \iint\limits_F |\frac{1}{2}| dudv \\ &= \frac{1}{2} \iint\limits_F dudv \\ &= \frac{1}{2}. \end{aligned}$$

□

10.6.2 计算下列围成的图像的面积:
(1)$(a_1x + b_1y + c_1)^2 + (a_2x + b_2y + c_2)^2 = 1$, 其中 $a_1b_2 \neq a_2b_1$;
(2)$\sqrt{x} + \sqrt{y} = \sqrt{a}$,$x = 0$ 和 $y = 0$.

解. (1) 令

$$\begin{cases} u = a_1x + b_1y + c_1 \\ v = a_2x + b_2y + c_2, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{a_1b_2 - a_2b_1},$$

对应的积分区域化为

$$F = \{(u,v) \in \mathbb{R}^2 | u^2 + v^2 \leq 1\},$$

带入由换元公式得到

$$\begin{aligned}\iint\limits_D dxdy &= \iint\limits_F |\frac{1}{a_1b_2 - a_2b_1}| dudv \\ &= \frac{\pi}{|a_1b_2 - a_2b_1|}.\end{aligned}$$

(2) 令

$$\begin{cases} x = r^2 \cos^4 \theta \\ y = r^2 \sin^4 \theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} 2r\cos^4\theta & -4r^2\cos^3\theta\sin\theta \\ 2r\sin^4\theta & 4r^2\sin^3\theta\cos\theta \end{vmatrix} = 8r^3\sin^3\theta\cos^3\theta,$$

对应的积分区域化为

$$F = [0, \sqrt{a}] \times [0, \frac{\pi}{2}],$$

带入由换元公式得到

$$\begin{aligned}\iint\limits_D dxdy &= \int_0^{\sqrt{a}} dr \int_0^{\frac{\pi}{2}} 8r^3 \sin^3\theta\cos^3\theta d\theta \\ &= \frac{a^2}{6}.\end{aligned}$$

□

10.6.3 求证:

$$\iint\limits_{|x|+|y|\leq 1} f(x+y)dxdy = \int_{-1}^1 f(t)dt.$$

解. (1) 令

$$\begin{cases} u = x + y \\ v = x - y, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2 \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2},$$

对应的积分区域化为

$$F = [-1,1] \times [-1,1],$$

带入由换元公式得到

$$\begin{aligned}\iint\limits_{|x|+|y|\leq 1} f(x+y)dxdy &= \iint\limits_{[-1,1]^2} f(u)|\frac{1}{2}|dudv \\ &= \int_{-1}^{1} f(t)dt.\end{aligned}$$

□

10.6.5 计算下列二重积分:

(2) $\iint\limits_{x^2+y^2\leq Rx} \sqrt{R^2-x^2-y^2}dxdy$;

(4) $\iint\limits_{x^2+y^2\leq x+y} \sqrt{x^2+y^2}dxdy$.

解. (2) 令

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r,$$

对应的积分区域化为

$$\{(r,\theta)\in\mathbb{R}^2|0\leq r\leq R\cos\theta, -\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}\},$$

带入由换元公式得到

$$\begin{aligned}\iint\limits_{x^2+y^2\leq Rx} \sqrt{R^2-x^2-y^2}dxdy &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{R\cos\theta} \sqrt{R^2-r^2}rdr \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3}(R^3-R^3|\sin^3\theta|)d\theta \\ &= (\frac{\pi}{3}-\frac{4}{9})R^3.\end{aligned}$$

(4) 令

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r,$$

对应的积分区域化为

$$\{(r,\theta)\in\mathbb{R}^2|0\leq r\leq (\cos\theta+\sin\theta), -\frac{\pi}{4}\leq\theta\leq\frac{3\pi}{4}\},$$

带入由换元公式得到

$$\iint\limits_{x^2+y^2\le x+y} \sqrt{x^2+y^2}dxdy = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta \int_0^{\cos\theta+\sin\theta} r^2 dr$$
$$= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{(\cos\theta+\sin\theta)^3}{3} d\theta$$
$$= \frac{8\sqrt{2}}{9}.$$

□

10.6.7 设常数 $a, b > 0$,

$$D = \{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\ x \ge y \ge 0\}.$$

计算二重积分

$$\iint_D \sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}} dxdy.$$

解. 令

$$\begin{cases} x = ar\cos\theta \\ y = br\sin\theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr,$$

对应的积分区域化为

$$\{(r,\theta) \in \mathbb{R}^2 | 0 \le r \le 1, 0 \le \theta \le \arctan\frac{a}{b}\},$$

带入由换元公式得到

$$\iint\limits_D \sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}} dxdy = \int_0^{\arctan\frac{a}{b}} d\theta \int_0^1 abr^2 dr$$
$$= \frac{1}{3} ab \arctan\frac{a}{b}.$$

□

12 Week 12

10.7.1 计算以下积分:

(1) $\int_V xyz d\mu$,V 为球体 $x^2+y^2+z^2 \le 1$ 在第一卦限中的部分;

(2) $\int_V (x+y+z) d\mu$,V 为平面 $x+y+z=1$ 和上和坐标平面所围成的立体;

(3) $\int_V xy^2z^3 dxdydz$,V 由 $z = xy$ 和 $z = 0$ 以及两张平面 $x = 1$ 和 $x = y$ 围成.

解. (1) 令

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta,$$

对应的积分区域化为

$$D = [0,1]\times[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}],$$

带入由换元公式得到

$$\begin{aligned} \int_V xyzd\mu &= \iiint_D r^5\sin^3\theta\cos\theta\sin\phi\cos\phi drd\theta d\phi \\ &= \frac{1}{48}. \end{aligned}$$

(2)

$$\begin{aligned} \int_V (x+y+z)d\mu &= \int_V xd\mu + \int_V yd\mu + \int_V zd\mu \\ &= 3\int_V xd\mu \\ &= 3\int_0^1 xdx \iint_{\substack{y+z\le 1-x \\ y,z\ge 0}} dydz \\ &= 3\int_0^1 \frac{1}{2}x(1-x)^2dx \\ &= \frac{1}{8}. \end{aligned}$$

(3)

$$\begin{aligned} \int_V xy^2z^3d\mu &= \int_0^1 xdx\int_0^x y^2dy\int_0^{xy} z^3dz \\ &= \frac{1}{364}. \end{aligned}$$

□

10.7.2 计算下列曲面围成的立体的体积:
(1)$z^2 = x^2 + \frac{y^2}{4}, 2z = x^2 + \frac{y^2}{4}$;
(2)$x^2 + y^2 = a^2, |x| + |y| = a$,

解. (1)

$$V = \iint_{x^2+\frac{y^2}{4}\le 4} (\sqrt{x^2+\frac{y^2}{4}} - \frac{x^2}{2} - \frac{y^2}{8})dxdy$$

令

$$\begin{cases} x = r\cos\theta \\ y = 2r\sin\theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ 2\sin\theta & 2r\cos\theta \end{vmatrix} = 2r,$$

对应的积分区域化为

$$D = [0,2] \times [0,2\pi],$$

带入由换元公式得到

$$\iint\limits_{x^2+\frac{y^2}{4}\le 4} (\sqrt{x^2+\frac{y^2}{4}} - \frac{x^2}{2} - \frac{y^2}{8})dxdy = \int_0^{2\pi} d\theta \int_0^2 (r - \frac{r^2}{2})2rdr \quad = \frac{8\pi}{3}.$$

(2)

$$\begin{aligned} V &= 8 \iint\limits_{\substack{x^2+y^2\le a^2 \\ x,z\ge 0}} dxdy \int_0^{a-x} dy \\ &= 8 \iint\limits_{\substack{x^2+y^2\le a^2 \\ x,z\ge 0}} (a-x)dxdz \\ &= 8 \iint\limits_{[0,a]\times[0,2\pi]} (a - r\cos\theta)rdrd\theta \\ &= (2\pi - \frac{8}{3})a^3. \end{aligned}$$

□

10.7.3 设 f 为连续函数, 求极限:

$$\lim_{r\to 0} \frac{3}{4\pi r^3} \iiint\limits_{x^2+y^2+z^2\le r^2} f(x,y,z)dxdydz.$$

解. 由积分中值定理得

$$\begin{aligned} \lim_{r\to 0} \frac{3}{4\pi r^3} \iiint\limits_{x^2+y^2+z^2\le r^2} f(x,y,z)dxdydz &= \lim_{r\to 0} \frac{3}{4\pi r^3} f(\xi,\eta,\phi) \frac{4\pi r^3}{3} \\ &= \lim_{r\to 0} f(\xi,\eta,\phi) \\ &= f(0,0,0). \end{aligned}$$

□

10.7.4 计算下列积分:

(1) $\iiint\limits_{x^2+y^2+z^2\le 2} \sqrt{x^2+y^2+z^2}dxdydz;$

(2) $\iiint\limits_{D} (x^2+y^2)dxdydz$, 其中 $D = \{(x,y,z) : z \ge 0, a^2 \le x^2+y^2+z^2 \le b^2\};$

(3) $\iiint\limits_D \left(1-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\right)^{1/2} dxdydz$, 其中 D 为椭球面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 包围的立体.

解. (1) 令

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta,$$

对应的积分区域化为

$$D = [0,2]\times[0,\pi]\times[0,2\pi],$$

带入由换元公式得到

$$\begin{aligned} \iiint\limits_{x^2+y^2+z^2\le 2} \sqrt{x^2+y^2+z^2}\,dxdydz &= \iiint\limits_D r^3\sin\theta drd\theta d\phi \\ &= 4\pi. \end{aligned}$$

(2) 令

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta,$$

对应的积分区域化为

$$E = [a,b]\times[0,\frac{\pi}{2}]\times[0,2\pi],$$

带入由换元公式得到

$$\begin{aligned} \iiint\limits_D (x^2+y^2)dxdydz &= \iiint\limits_E r^4\sin^3\theta drd\theta d\phi \\ &= \frac{4(b^5-a^5)\pi}{15}. \end{aligned}$$

(3) 令

$$\begin{cases} x = ar\sin\theta\cos\phi \\ y = br\sin\theta\sin\phi \\ z = cr\cos\theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix} a\sin\theta\cos\phi & ar\cos\theta\cos\phi & -ar\sin\theta\cos\phi \\ b\sin\theta\sin\phi & br\cos\theta\sin\phi & br\sin\theta\cos\phi \\ c\cos\theta & -cr\sin\theta & 0 \end{vmatrix}=abcr^2\sin\theta,$$

对应的积分区域化为

$$E=[0,1]\times[0,\frac{\pi}{2}]\times[0,2\pi],$$

带入由换元公式得到

$$\begin{aligned}\iiint\limits_D \Big(1-\Big(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\Big)\Big)^{1/2}dxdydz &= abc\iiint\limits_E r^2\sqrt{1-r^2}\sin\theta drd\theta d\phi \\ &= \frac{abc\pi^2}{4}.\end{aligned}$$

□

10.7.5 计算由下列曲面围成的立体的体积:

(1)$a_ix+b_iy+c_iz=\pm h_i(i=1,2,3)$, 设行列式

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\neq 0;$$

(4)$(x^2+y^2+z^2)^n=z^{2n-1}(n\in\mathbf{N}^*)$;

(5)$\Big(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\Big)^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}$.

解. (1) 令

$$\begin{cases} u=a_1x+b_1y+c_1z \\ v=a_2x+b_2y+c_2z \\ w=a_3x+b_3y+c_3z, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

对应的积分区域化为

$$E=[-h_1,h_1]\times[-h_2,h_2]\times[-h_3,h_3],$$

带入由换元公式得到

$$\begin{aligned}V=\iiint\limits_E \left\|\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^{-1}\right\| dudvdw \\ =\frac{8h_1h_2h_3}{\left\|\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}\right\|}.\end{aligned}$$

(4) 令

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta,$$

对应的积分区域化为

$$E = \{(r,\theta,\phi)|0 \le r \le \cos^{2n-1}\theta, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le 2\pi\},$$

带入由换元公式得到

$$\begin{aligned} V &= \iiint\limits_E r^2\sin\theta dr d\theta d\phi \\ &= \frac{\pi}{3(3n-1)}. \end{aligned}$$

(5) 令

$$\begin{cases} x = ar\sin\theta\cos\phi \\ y = br\sin\theta\sin\phi \\ z = cr\cos\theta, \end{cases}$$

换元的 Jcaobi 行列式为

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} a\sin\theta\cos\phi & ar\cos\theta\cos\phi & -ar\sin\theta\cos\phi \\ b\sin\theta\sin\phi & br\cos\theta\sin\phi & br\sin\theta\cos\phi \\ c\cos\theta & -cr\sin\theta & 0 \end{vmatrix} = abcr^2\sin\theta,$$

对应的积分区域化为

$$E = \{(r,\theta,\phi)|0 \le r^2 \le \sin^2\theta, 0 \le \theta \le \pi, 0 \le \phi \le 2\pi\},$$

带入由换元公式得到

$$\begin{aligned} V &= \iiint\limits_E abcr^2\sin\theta dr d\theta d\phi \\ &= 8abc\int_0^{\frac{\pi}{2}}\sin\theta d\theta\int_0^{\sin\theta}r^2dr\int_0^{\frac{\pi}{2}}d\phi \quad = \frac{\pi^2}{4}abc. \end{aligned}$$

□

10.8.1 计算下列 n 重积分：

(1) $\int\limits_{[0,1]^n}\cdots\int(x_1^2+...+x_n^2)dx_1...dx_n$;

(2) $\int\limits_{[0,1]^n}\cdots\int(x_1+...+x_n)^2dx_1...dx_n$.

解. (1)

$$\begin{aligned}\int\cdots\int_{[0,1]^n}(x_1^2+...+x_n^2)dx_1...dx_n &= n\int\cdots\int_{[0,1]^n} x_1^2 dx_1...dx_n\\ &= n\int_0^1 x_1^2 dx_1\int\cdots\int_{[0,1]^{n-1}} dx_2...dx_n\\ &= n\int_0^1 x_1^2 dx_1\\ &= \frac{n}{3}.\end{aligned}$$

(2)

$$\begin{aligned}\int\cdots\int_{[0,1]^n}(x_1+...+x_n)^2 dx_1...dx_n &= \int\cdots\int_{[0,1]^n}\sum_{k=1}^n x_k^2+\sum_{i\neq j}x_ix_jdx_1...dx_n\\ &= n\int\cdots\int_{[0,1]^n} x_1^2dx_1...dx_n+\frac{n(n-1)}{2}\int\cdots\int_{[0,1]^n} x_1x_2dx_1...dx_n\\ &= \frac{n}{3}+\frac{n(n-1)}{4}\\ &= \frac{n^2}{4}+\frac{n}{12}.\end{aligned}$$

□

10.8.2 计算累次积分:

$$\int_0^1 dx_1\int_0^{x_1}...\int_0^{x_{n-1}} x_1...x_{n-1}x_ndx_n.$$

解.

$$\begin{aligned}f_n=\int_0^1 dx_1\int_0^{x_1}...\int_0^{x_{n-1}} x_1...x_{n-1}x_ndx_n &= \int_0^1 x_1dx_1\int_0^{x_1}...\int_0^{x_{n-1}} x_2...x_ndx_n\\ &= \int_0^1 x_1x_1^{2n-2}dx_1\int_0^1 dx_2\int_0^{x_1}...\int_0^{x_{n-1}} x_2...x_{n-1}dx_{n-1}\\ &= \frac{1}{2n}f_{n-1}\\ &= \frac{1}{(2n)!!}\end{aligned}$$

□

10.8.3 计算下列 $\mathbb{R}^n$ 中区域的体积 $(a_1,a_2,...,a_n>0)$:

(1)$V_n=(x_1,x_2,...,x_n):\dfrac{x_1}{a_1}+\dfrac{x_2}{a_2}+...+\dfrac{x_n}{a_n}\leq 1, x_1,x_2,...,x_n\geq 0$;

(2)$V_n(a)=\left\{(x_1,x_2,...,x_n):|x_1|+|x_2|+...+|x_n|\leq a\right\}$.

解. (1)

$$
\begin{aligned}
f_n(a_1, ..., a_n) &= \int\limits_{\substack{\frac{x_1}{a_1}+...+\frac{x_n}{a_n}\le 1\\ xi\ge 0}} \cdots \int dx_1...dx_n \\
&= \int_0^{a_1} dx_1 \int\limits_{\substack{\frac{x_2}{a_2}+...+\frac{x_n}{a_n}\le 1-\frac{x_1}{a_1}\\ xi\ge 0}} \cdots \int dx_2...dx_n \\
&= \int_0^{a_1} \big(1-\frac{x_1}{a_1}\big)^{n-1} dx_1 f_{n-1}(a_2, ..., a_n) \\
&= \frac{a_1}{n} f_{n-1}(a_2, ..., a_n) \\
&= \frac{a_1...a_n}{n!}.
\end{aligned}
$$

(2)

$$
\begin{aligned}
f_n(a) &= \int\limits_{V_n(a)} d\mu \\
&= 2\int_0^a f_{n-1}(a-x_n)dx_n \\
&= 2\int_0^a (a-x_n)^{n-1} f_{n-1}(1)dx_n \\
&= \frac{2a}{n} f_{n-1}(a) \\
&= \frac{(2a)^n}{n!}.
\end{aligned}
$$

□

10.8.4 设 K 为二元连续函数, 对 $n \in \mathbf{N}^*$, 令

$$K_n(x, y) = \int\limits_{[a,b]^n} \cdots \int K(x, t_1)K(t_1, t_2)...K(t_n, y)dt_1...dt_n.$$

求证: 对任意 $m, n \in \mathbf{N}^*$, 有

$$K_{m+n+1}(x, y) = \int_a^b K_m(x, t)K_n(t, y)dt.$$

解.

$$
\begin{aligned}
&\int_a^b K_n(x, t)K_m(t, y)dt \\
&= \int_a^b \int\limits_{[a,b]^n} \cdots \int K(x, t_1)K(t_1, t_2)...K(t_n, t)dt_1...dt_n \int\limits_{[a,b]^m} \cdots \int K(t, t_{n+1})K(t_{n+1}, t_{n+2})...K(t_{n+m}, t)dt_{n+1}...dt_{n+m}dt \\
&= \int\limits_{[a,b]^{n+m}} \cdots \int K(x, t_1)K(t_1, t_2)...K(t_{n+m}, y)dt_1...dt_{n+m}.
\end{aligned}
$$

□

10.8.5 设 $a_1, a_2, ..., a_n > 0$,

$$V_n = \left\{(x_1, x_2, ..., x_n) : \frac{|x_i|}{a_2} + |\frac{|x_n|}{a_n} \le 1(i-1, 2, ..., n-1)\right\}.$$

求 V_n 的体积.

解.

$$\begin{aligned}\int_{V_n} d\mu &= 2^n \int_0^{a_n} dx_n \int_0^{a_1(1-\frac{x_n}{a_n})} dx_1 ... \int_0^{a_{n-1}(1-\frac{x_n}{a_n})} dx_{n-1} \\ &= 2^n a_1...a_{n-1} \int_0^{a_n} \big(1-\frac{x_n}{a_n}\big) dx_n \\ &= \frac{2^n}{n} a_1...a_n\end{aligned}$$

□

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11.1.1 $\int_\Gamma (x^2+y^2)^n \mathrm{d}s, \Gamma: x=a\cos t, y=a\sin t()0\le t\le 2\pi$.

解. $\mathrm{d}x=-a\sin t\mathrm{d}t, \mathrm{d}y=a\cos t\mathrm{d}t$, 故

$$\int_\Gamma (x^2+y^2)^n \mathrm{d}s, \Gamma: x=a\cos t, y=a\sin t()0\le t\le 2\pi = \int_0^{2\pi} a^{2n}|a|\mathrm{d}t = 2\pi|a|^{2n+1}$$

□

11.1.2 $\int_\Gamma (x+y)\mathrm{d}s, \Gamma:$ 顶点为 $(0,0),(1,0),(0,1)$ 的三角形的边界。

解.

$$\int_\Gamma (x+y)\mathrm{d}s = \int_0^1 x\mathrm{d}x + \int_1^0 -\sqrt{2}\mathrm{d}x + \int_1^0 y\mathrm{d}y = 1+\sqrt{2}$$

□

11.1.3 $\int_\Gamma z\mathrm{d}s, \Gamma:$ 圆锥螺线：$x=t\cos t, y=t\sin t, z=t(0\le t\le 2\pi)$.

解.

$$\begin{aligned}&\mathrm{d}x=(\cos t - t\sin t)\mathrm{d}t, \mathrm{d}y=(\sin t+t\cos t)\mathrm{d}t, \mathrm{d}z=\mathrm{d}t. \\ &\mathrm{d}s=\sqrt{\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2}=\sqrt{2+t^2}\mathrm{d}t. \\ &\int_\Gamma z\mathrm{d}s = \int_0^{2\pi} t\sqrt{2+t^2}\mathrm{d}t = \frac{1}{3}\big((2+4\pi^2)^{\frac{3}{2}}-2\sqrt{2}\big).\end{aligned}$$

□

11.1.4 $\int_\Gamma x^2\mathrm{d}s, \Gamma:$ 圆周 $x^2+y^2+z^2=a^2, x+y+z=0$.

解.

$$\int_\Gamma x^2\mathrm{d}s = \frac{1}{3}\int_\Gamma x^2+y^2+z^2\mathrm{d}s = \frac{a^2}{3}\int_\Gamma \mathrm{d}s = \frac{2}{3}\pi a^3.$$

□

11.1.5 $\int_\Gamma y^2\mathrm{d}s, \Gamma:$ 旋轮线的一拱，$x=a(t-\sin t), y=a(1-\cos t)(0\le t\le 2\pi)$.

解.

$$\mathrm{d}x = a(1-\cos t)\mathrm{d}t, \mathrm{d}y = a\sin t\mathrm{d}t.$$

$$\int_\Gamma y^2\mathrm{d}s = \int_\Gamma a^2(1-\cos t)^2 2|a\sin\frac{t}{2}|\mathrm{d}t = 8|a|^3\int_0^{2\pi}\sin^5\frac{t}{2}\mathrm{d}t = \frac{256}{15}|a|^3.$$

□

11.2.1 计算第二型曲线积分。

(1) $\int_\Gamma \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2+y^2}, \Gamma$ 表示逆时针方向的圆周 $x^2+y^2=a^2$;

解. $x = a\cos t, y = a\sin t$, 故 $\mathrm{d}x = -a\sin t\mathrm{d}t, \mathrm{d}y = a\cos t\mathrm{d}t$, 方向为 t 增加的方向,

$$\frac{x\mathrm{d}y - y\mathrm{d}x}{x^2+y^2} = \int_0^{2\pi}\frac{a^2\cos^2 t\mathrm{d}t + a^2\sin^2 t\mathrm{d}t}{a^2} = 2\pi.$$

□

(3) $\int_\Gamma (x^2-2xy)\mathrm{d}x + (y^2-2xy)\mathrm{d}y, \Gamma: x = y^2(-1\le y\le 1)$, 沿 y 增加的方向;

解.

$$\int_\Gamma (x^2-2xy)\mathrm{d}x + (y^2-2xy)\mathrm{d}y = \int_{-1}^1 (y^4-2y^3)2y\mathrm{d}y + (y^2-2y^3)\mathrm{d}y = -\frac{14}{15}.$$

□

(5) $\int_\Gamma (x^2+y^2)\mathrm{d}y, \Gamma$ 是直线 $x=1, x=3$ 和 $y=1, y=4$ 构成的矩形，沿逆时针方向。

解.

$$\int_\Gamma (x^2+y^2)\mathrm{d}y = \int_1^4 (9+y^2)\mathrm{d}y + \int_4^1 (1+y^2)\mathrm{d}y = 24.$$

□

11.2.2 设常数 a, b, c 满足 $ac-b^2>0$. 计算 $\int_\Gamma \frac{x\mathrm{d}y - y\mathrm{d}x}{ax^2+2bxy+cy^2}$, 其中 Γ 为逆时针方向的单位圆周。

解. $x=\cos t, y=\sin t$, 故 $\mathrm{d}x = -\sin t\mathrm{d}t, \mathrm{d}y = \cos t\mathrm{d}t$.

$$\begin{aligned}\int_\Gamma \frac{x\mathrm{d}y - y\mathrm{d}x}{ax^2+2bxy+cy^2} &= \int_0^{2\pi}\frac{\mathrm{d}t}{a\cos^2 t + 2b\cos t\sin t + c\sin^2 t}\\ &= 2\int_{-\pi/2}^{\pi/2}\frac{1}{a+2b\tan t + c\tan^2 t}\cdot\frac{\mathrm{d}t}{\cos^2 t}\\ &= 2\int_{-\infty}^{+\infty}\frac{1}{a+2bt+ct^2}\mathrm{d}t\\ &= \frac{2\pi c}{|c|\sqrt{ac-b^2}}.\end{aligned}$$

□

11.2.3 计算第二型曲线积分，曲线的正向是参数增加的方向;

(1) $\int_\Gamma xz^2\mathrm{d}x + yx^2\mathrm{d}y + zy^2\mathrm{d}z, \Gamma: x=t, y=t^2, z=t^3(0\le t\le 1)$;

解.

$$\int_\Gamma xz^2\mathrm{d}x + yx^2\mathrm{d}y + zy^2\mathrm{d}z = \int_0^1 t^7 + 2t^5 + 3t^9\mathrm{d}t = \frac{91}{120}.$$

□

(2) $\int_\Gamma (y+z)\mathrm{d}x+(z+x)\mathrm{d}y+(x+y)\mathrm{d}z, \Gamma : x = a\sin^2 t, y = 2a\sin t\cos t, z = a\cos^2 t(0 \le t \le \pi)$.

解. $\mathrm{d}x = 2a\sin t\cos t\mathrm{d}t, \mathrm{d}y = 2a(\cos^2 t - \sin^2 t), \mathrm{d}z = -2a\sin t\cos t.$

$$\begin{aligned}&\int_\Gamma (y+z)\mathrm{d}x + (z+x)\mathrm{d}y + (x+y)\mathrm{d}z\\ =&a^2\int_0^\pi (2\sin t\cos t + \cos^2 t)\cdot 2\sin t\cos t + 2(\cos^2 t - \sin^2 t) - 2\sin t\cos t(\sin^2 t + 2\sin t\cos t)\mathrm{d}t\\ =&a^2\int_0^\pi \cos 2t(2+\sin 2t)\mathrm{d}t\\ =&0\end{aligned}$$

□

11.2.4 $\int_\Gamma (y^2-z^2)\mathrm{d}x+(z^2-x^2)\mathrm{d}y+(x^2-y^2)\mathrm{d}z, \Gamma$ 为球面片 $x^2+y^2+z^2=1, x\ge 0, y\ge 0, z\ge 0$ 的边界，方向是 $(1,0,0)\to(0,1,0)\to(0,0,1)\to(1,0,0)$.

解. 设题中定义的曲线 $(1,0,0)\to(0,1,0)$ 段为 Γ_1,$(0,1,0)\to(0,0,1)$ 段为 Γ_2,$(0,0,1)\to(1,0,0)$ 段为 Γ_3.

$$\int_{\Gamma_1} (y^2-z^2)\mathrm{d}x + (z^2-x^2)\mathrm{d}y + (x^2-y^2)\mathrm{d}z = \int_{\Gamma_1} y^2\mathrm{d}x - x^2\mathrm{d}y = \int_0^{\pi/2} -\sin^3 t - \cos^3 t\mathrm{d}t = -\frac{4}{3}$$

同理，有

$$\int_{\Gamma_2} (y^2-z^2)\mathrm{d}x+(z^2-x^2)\mathrm{d}y+(x^2-y^2)\mathrm{d}z = \int_{\Gamma_3} (y^2-z^2)\mathrm{d}x+(z^2-x^2)\mathrm{d}y+(x^2-y^2)\mathrm{d}z = \int_{\Gamma_1} (y^2-z^2)\mathrm{d}x+(z^2-x^2)\mathrm{d}y+(x^2-$$

因此

$$\int_\Gamma (y^2-z^2)\mathrm{d}x + (z^2-x^2)\mathrm{d}y + (x^2-y^2)\mathrm{d}z = -4$$

□

11.2.5 证明: $\left|\int_\Gamma \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{p}\right| \le \int_\Gamma ||\boldsymbol{F}||\mathrm{d}s.$

解.

$$\left|\int_\Gamma \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{p}\right| = \left|\int_\Gamma \sum_{i=1}^n F_i\cdot\mathrm{d}p_i\right| \le \int_\Gamma \sum_{i=1}^n |F_i|\cdot|\mathrm{d}p_i| \le \int_\Gamma \left(\sum_{i=1}^n |F_i|^2\right)^{\frac12}\left(\sum_{i=1}^n |\mathrm{d}p_i|^2\right)^{\frac12} = \int_\Gamma ||\boldsymbol{F}||\mathrm{d}s$$

□

11.3.1 利用 Green 公式计算

(1) $\int_\Gamma xy^2\mathrm{d}y - x^2y\mathrm{d}x, \Gamma$ 为圆周 $x^2+y^2=a^2$, 按逆时针方向；

(2) $\int_\Gamma (x+y)\mathrm{d}x - (x-y)\mathrm{d}y, \Gamma$ 为椭圆 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, 按逆时针方向；

(3) $\int_\Gamma e^x\cos y\mathrm{d}x + e^x\cos y\mathrm{d}y, \Gamma$ 为上半圆周 $x^2+y^2=ax$ 沿 x 增加的方向。

解. (1)$P(x,y)=-x^2y, Q(x,y)=xy^2$,

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=y^2+x^2$$

由 Green 公式

$$\int_\Gamma xy^2\mathrm{d}y-x^2y\mathrm{d}x=\iint_{B_a(0)}x^2+y^2\mathrm{d}x\mathrm{d}y=\int_0^{2\pi}\int_0^a r^3\mathrm{d}r\mathrm{d}\theta=\frac{\pi a^4}{2}.$$

(2)$P(x,y)=x+y, Q(x,y)=y-x$,

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=-2$$

由 Green 公式

$$\int_\Gamma (x+y)\mathrm{d}x-(x-y)\mathrm{d}y=\iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1}-2\mathrm{d}x\mathrm{d}y=-2\pi ab.$$

(3) 设 $\Gamma_1=-\Gamma, \Gamma_2=\{(x,0)|x:0\to a\}$, 则 $P(x,y)=e^x\sin y, Q(x,y)=e^x\cos y$,

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$$

由 Green 公式

$$\int_{\Gamma_1+\Gamma_2}e^x\cos y\mathrm{d}x+e^x\cos y\mathrm{d}y=\iint_{x^2+y^2\le ax, y\ge 0}0\mathrm{d}x\mathrm{d}y=0,$$

又在 Γ_2 上 $y=0, \mathrm{d}y=0$ 故

$$\int_{\Gamma_2}e^x\cos y\mathrm{d}x+e^x\cos y\mathrm{d}y=0.$$

因此

$$\int_\Gamma e^x\cos y\mathrm{d}x+e^x\cos y\mathrm{d}y=-\int_{\Gamma_1}e^x\cos y\mathrm{d}x+e^x\cos y\mathrm{d}y=-\int_{\Gamma_1+\Gamma_2}e^x\cos y\mathrm{d}x+e^x\cos y\mathrm{d}y=0.$$

□

11.3.2 用 Green 公式计算面积。

(1) 星形线 $x=a\cos^3 t, y=a\sin^3 t(0\le t\le 2\pi)$;

(2) 双纽线 $(x^2+y^2)^2=a^2(x^2-y^2)$;

(3) Descartes 叶形线 $x^3+y^3=3axy(a>0)$.

解.

$$\sigma(D)=\int_{\partial D}x\mathrm{d}y=-\int_{\partial D}y\mathrm{d}x=\frac{1}{2}\int_{\partial D}x\mathrm{d}y-y\mathrm{d}x.$$

(1)$\mathrm{d}x=-3a\cos^2 t\sin t\mathrm{d}t, \mathrm{d}y=3a\cos t\sin^2 t\mathrm{d}t$,

$$\begin{aligned}\sigma(D)&=\frac{1}{2}\int_{\partial D}x\mathrm{d}y-y\mathrm{d}x\\&=\frac{1}{2}\int_{\partial D}(3a^2\cos^4 t\sin^2 t+3a^2\cos^2 t\sin^4 t)\mathrm{d}t\\&=\frac{3a^2}{2}\int_{\partial D}\cos^2 t\sin^2 t\mathrm{d}t\\&=\frac{3\pi a^2}{8}\end{aligned}$$

(2) 令 $y=x\tan\theta$, 代入方程得 $\big(x^2(1+\tan^2\theta)\big)^2=a^2(1-\tan^2\theta)x^2$,

$$x^2=\frac{a^2(1-\tan^2\theta)}{(1+\tan^2\theta)^2}=a^2\cos^2\theta\cos(2\theta),y^2=a^2\sin^2\theta\cos(2\theta)$$

θ 范围：由计算知 $\cos(2\theta)\geq 0$, 故 $\theta\in[0,\frac{\pi}{4}]\cup[\frac{3\pi}{4},\frac{5\pi}{4}]\cup[\frac{7}{4\pi},2\pi]$. 由对称性，只考虑第一象限即可。

$$\begin{aligned}y\mathrm{d}x=x\tan\theta\mathrm{d}x=&a^2\tan\theta\big(-2\cos\theta\sin\theta\cos(2\theta)-2\cos^2\theta\sin(2\theta)\big)\mathrm{d}\theta\\=&-2a^2\big(\sin^2\theta\cos(2\theta)+\sin\theta\cos\theta\sin(2\theta)\big)\mathrm{d}\theta\\x\mathrm{d}y=\frac{y}{\tan\theta}\mathrm{d}y=&\frac{a^2}{\tan\theta}\big(2\sin\theta\cos\theta\cos(2\theta)-2\sin^2\theta\sin(2\theta)\big)\mathrm{d}\theta\\=&2a^2\big(\cos^2\theta\cos(2\theta)-\sin\theta\cos\theta\sin(2\theta)\big)\mathrm{d}\theta\end{aligned}$$

代入得

$$\frac{1}{4}\sigma(D)=\frac{1}{8}\int_{\partial D}x\mathrm{d}y-y\mathrm{d}x=\frac{1}{8}\int_0^{\pi/4}2a^2\cos(2\theta)\mathrm{d}\theta=\frac{a^2}{4},$$

故

$$\sigma(D)=a^2.$$

(3) 令 $y=xt$, 代入方程得 $x^3(1+t^3)=3ax^2t$, 得到参数方程

$$x=\frac{3at}{1+t^3},y=\frac{3at^2}{1+t^3},$$

用参数 $u=\frac{1}{t}$ 代换可得上述 x,y 交换的表达式，容易知道上述用 t 作为参数的表达式在 $t\in[0,1]$ 时是从 $(0,0)$ 到 $\left(\frac{3a}{2},\frac{3a}{2}\right)$ 的曲线 Γ_1，而在参数 $u\in[0,1]$ 时是与 Γ_1 关于 $y=x$ 对称的曲线 Γ_2，因此 Γ_1 和 $-\Gamma_2$ 构成了封闭曲线，其参数范围 $t\in[0,+\infty)$.

$$\mathrm{d}x=\frac{3a(1-2t^3)}{1+t^3)^2}\mathrm{d}t$$

由 Green 公式，换元 $u=t^3$,

$$\begin{aligned}\sigma(D)=-\int_{\partial}y\mathrm{d}x=9a^2\int_0^{+\infty}\frac{t^2(2t^3-1)}{(1+t^3)^3}\mathrm{d}t&=3a^2\int_0^{+\infty}\frac{2u-1}{(1+u)^3}\mathrm{d}u\\&=3a^2\int_0^{+\infty}\frac{2}{(1+u)^2}-\frac{3}{(1+u)^3}\mathrm{d}u\\&=\frac{3a^2}{2}.\end{aligned}$$

□

11.3.3 封闭曲线 $\Gamma:x=\varphi(t),y=\psi(t)(\alpha\leq t\leq\beta)$, 参数增加的方向是正方向，求证 Γ 围城的面积

$$A=\frac{1}{2}\int_\alpha^\beta\begin{vmatrix}\varphi(t)&\psi(t)\\\varphi'(t)&\psi'(t)\end{vmatrix}\mathrm{d}t.$$

解.

$$\mathrm{d}x = \varphi'(t)\mathrm{d}t, \mathrm{d}y = \psi'(t)\mathrm{d}t$$

故

$$A = \frac{1}{2}\int_{\partial D} x\mathrm{d}y - y\mathrm{d}x = \frac{1}{2}\int_{\alpha}^{\beta} \begin{vmatrix} \varphi(t) & \psi(t) \\ \varphi'(t) & \psi'(t) \end{vmatrix} \mathrm{d}t.$$

□

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11.3.4 $f \in C^1, \Gamma$ 是任意一条分段光滑的封闭曲线。证明:

(1) $\int_\Gamma f(xy)(y\mathrm{d}x + x\mathrm{d}y) = 0;$

(2) $\int_\Gamma f(x^2 + y^2)(x\mathrm{d}x + y\mathrm{d}y) = 0;$

(3) $\int_\Gamma f(x^n + y^n)(x^{n-1}\mathrm{d}x + y^{n-1}\mathrm{d}y) = 0.$

解. (1) $P(x,y) = f(xy)y, Q(x,y) = f(xy)x,$

$$\frac{\partial Q}{\partial x} = f(xy) + yf'(xy)x, \frac{\partial P}{\partial y} = f(xy) + xf'(xy)y$$

故

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

由 Green 公式知 $\int_\Gamma f(xy)(y\mathrm{d}x + x\mathrm{d}y) = 0.$

(3) $P(x,y) = f(x^n + y^n)x^{n-1}, Q(x,y) = f(x^n + y^n)y^{n-1},$

$$\frac{\partial Q}{\partial x} = y^{n-1}f'(x^n + y^n)nx^{n-1}, \frac{\partial P}{\partial y} = x^{n-1}f'(x^n + y^n)ny^{n-1},$$

故

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

由 Green 公式知 $\int_\Gamma f(x^n + y^n)(x^{n-1}\mathrm{d}x + y^{n-1}\mathrm{d}y) = 0.$ □

11.3.5 Γ 是 $\mathbb{R}^2$ 中的光滑封闭曲线，$\boldsymbol{n}$ 是单位外法向量，设 $\boldsymbol{a}$ 是一个固定的单位向量，求证:

$$\int_\Gamma \cos(\boldsymbol{a}, \boldsymbol{n})\mathrm{d}s = 0.$$

解. 设 $\boldsymbol{t}$ 是弧长参数下的切向量，则 $\boldsymbol{t}\mathrm{d}s = (\mathrm{d}x, \mathrm{d}y)$, 又有 $||\boldsymbol{t}|| = ||\boldsymbol{n}||$ 和 $\boldsymbol{n} \perp \boldsymbol{t}$, 故 $\boldsymbol{n}\mathrm{d}s = \pm(\mathrm{d}y, -\mathrm{d}x)$, 由 Green 公式

$$\int_\Gamma \cos(\boldsymbol{a}, \boldsymbol{n})\mathrm{d}s = \pm\int_\Gamma a_1\mathrm{d}y - a_2\mathrm{d}x = 0.$$

□

11.3.6 Γ 是光滑封闭曲线，$\boldsymbol{n}$ 是单位外法向量，计算

$$\int_\Gamma x\cos(\boldsymbol{n}, \boldsymbol{i}) + y\cos(\boldsymbol{n}, \boldsymbol{j})\mathrm{d}s.$$

解.

$$\int_\Gamma x\cos(\boldsymbol{n},\boldsymbol{i})+y\cos(\boldsymbol{n},\boldsymbol{j})\mathrm{d}s=\int_\Gamma x\boldsymbol{n}\cdot\boldsymbol{i}+y\boldsymbol{n}\cdot\boldsymbol{j}\mathrm{d}s=\int_\Gamma x\mathrm{d}y-y\mathrm{d}x=2\sigma(\Omega).$$

□

11.3.7 (1) $\int_L(x^2+2xy-y^2)\mathrm{d}x+(x^2-2xy-y^2)\mathrm{d}y, L$ 是连接 $A=(0,0),B=(2,1)$ 的任意光滑线段；

(2) $\int_L\frac{(x+y)\mathrm{d}x-(x-y)\mathrm{d}y}{x^2+y^2}$, 其中 L 是位于上半平面从点 $(-1,0)$ 到 $(1,0)$ 的任意光滑线段。

解. (1)$P(x,y)=x^2+2xy-y^2, Q(x,y)=x^2-2xy-y^2$.

$$\frac{\partial Q}{\partial x}=2(x-y),\frac{\partial P}{\partial y}=2(x-y)$$

故该积分与积分路径无关。选择道路为 $(0,0)\to(2,0)\to(2,1)$ 的折线，则

$$\int_L(x^2+2xy-y^2)\mathrm{d}x+(x^2-2xy-y^2)\mathrm{d}y=\int_0^2x^2\mathrm{d}x+\int_0^1(4-4y-y^2)\mathrm{d}y=\frac{13}{3}$$

(2)$P(x,y)=\frac{x+y}{x^2+y^2}, Q(x,y)=\frac{y-x}{x^2+y^2}$.

$$\frac{\partial Q}{\partial x}=\frac{x^2-2xy-y^2}{(x^2+y^2)^2},\frac{\partial P}{\partial y}=\frac{x^2-2xy-y^2}{(x^2+y^2)^2}$$

故该积分与积分路径无关。选择道路为 $(-1,0)\to(1,0)$ 的单位圆弧（上半平面内），则

$$\int_L\frac{(x+y)\mathrm{d}x-(x-y)\mathrm{d}y}{x^2+y^2}=\int_{-\pi/2}^{\pi/2}-(\cos\theta+\sin\theta)\sin\theta\mathrm{d}\theta-(\cos\theta-\sin\theta)\cos\theta\mathrm{d}\theta=\pi$$

□

11.3.8 计算

$$\int_\Gamma\frac{e^x}{x^2+y^2}\big((x\sin y-y\cos y)\mathrm{d}x+(x\cos y+y\sin y)\mathrm{d}y\big),$$

其中 Γ 是原点在其内部的分段光滑的闭曲线。

解. 设 Γ_ε 为 Γ 围成的区域去掉小圆盘 $B_\varepsilon(0)$ 所得区域的边界。则

$$\begin{aligned}&\int_\Gamma\frac{e^x}{x^2+y^2}\big((x\sin y-y\cos y)\mathrm{d}x+(x\cos y+y\sin y)\mathrm{d}y\big)\\=&\int_{\Gamma_1}\frac{e^x}{x^2+y^2}\big((x\sin y-y\cos y)\mathrm{d}x+(x\cos y+y\sin y)\mathrm{d}y\big)\\+&\int_{\partial B_\varepsilon(0)}\frac{e^x}{x^2+y^2}\big((x\sin y-y\cos y)\mathrm{d}x+(x\cos y+y\sin y)\mathrm{d}y\big),\end{aligned}$$

$P(x,y)=\frac{e^x}{x^2+y^2}(x\sin y-y\cos y), Q(x,y)=\frac{e^x}{x^2+y^2}(x\cos y+y\sin y)$, 计算易得 $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$, 故

$$\int_{\Gamma_1}\frac{e^x}{x^2+y^2}\big((x\sin y-y\cos y)\mathrm{d}x+(x\cos y+y\sin y)\mathrm{d}y\big)=0,$$

$$
\begin{aligned}
&\int_{\partial B_\varepsilon(0)} \frac{e^x}{x^2+y^2}\big((x\sin y - y\cos y)\mathrm{d}x + (x\cos y + y\sin y)\mathrm{d}y\big) \\
=&\frac{1}{\varepsilon^2}\int_{\partial B_\varepsilon(0)} e^x\big((x\sin y - y\cos y)\mathrm{d}x + (x\cos y + y\sin y)\mathrm{d}y\big) \\
\overset{\text{Green}}{=}&\frac{1}{\varepsilon^2}\iint_{B_\varepsilon(0)} e^x 2\cos y \mathrm{d}x\mathrm{d}y \overset{\varepsilon\to 0+}{\longrightarrow} \frac{1}{\varepsilon^2}\cdot 2\pi\varepsilon^2 = 2\pi
\end{aligned}
$$

故

$$
\begin{aligned}
&\int_\Gamma \frac{e^x}{x^2+y^2}\big((x\sin y - y\cos y)\mathrm{d}x + (x\cos y + y\sin y)\mathrm{d}y\big) \\
=&\int_{\Gamma_1} \frac{e^x}{x^2+y^2}\big((x\sin y - y\cos y)\mathrm{d}x + (x\cos y + y\sin y)\mathrm{d}y\big) \\
+&\int_{\partial B_\varepsilon(0)} \frac{e^x}{x^2+y^2}\big((x\sin y - y\cos y)\mathrm{d}x + (x\cos y + y\sin y)\mathrm{d}y\big) \\
=&2\pi
\end{aligned}
$$

□

12.1.1 锥面 $z=\sqrt{x^2+y^2}$ 被圆柱面 $x^2+y^2=2x$ 截下的部分的面积。

解.

$$
\begin{aligned}
&\frac{\partial z}{\partial x} = \frac{x}{z}, \frac{\partial z}{\partial y} = \frac{y}{z} \\
&\mathrm{d}\sigma = \sqrt{1+\Big(\frac{\partial z}{\partial x}\Big)^2+\Big(\frac{\partial z}{\partial y}\Big)^2}\mathrm{d}x\mathrm{d}y = \sqrt{2}\mathrm{d}x\mathrm{d}y \\
&\sigma(D) = \iint\limits_{\substack{x^2+y^2\le 2x \\ z=\sqrt{x^2+y^2}}} \mathrm{d}\sigma = \iint\limits_{x^2+y^2\le 2x} \sqrt{2}\mathrm{d}x\mathrm{d}y = \sqrt{2}\pi
\end{aligned}
$$

□

12.1.3 圆柱面 $x^2+y^2=a^2$ 介于平面 $x\pm z=0$ 之间的部分的面积。

解.

$$
\begin{aligned}
&\boldsymbol{r} = (a\cos\theta, a\sin\theta, z) \\
&\text{if } x\ge 0, -x\le z\le x, -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}. \\
&\boldsymbol{r}_\theta = (-a\sin\theta, a\cos\theta, 0), \boldsymbol{r}_z = (0,0,1) \\
&\boldsymbol{r}_\theta\times\boldsymbol{r}_z = (a\cos\theta, a\sin\theta, 0), ||\boldsymbol{r}_\theta\times\boldsymbol{r}_z|| = a, \\
&\sigma(D) = \iint\limits_{\substack{x^2+y^2=a^2 \\ -|x|\le z\le |x|}} \mathrm{d}\sigma = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-a\cos\theta}^{a\cos\theta} a\mathrm{d}z\mathrm{d}\theta = 4a^2
\end{aligned}
$$

□

12.1.5 马鞍面 $az=xy$ 被圆柱面 $x^2+y^2=a^2$ 截下的部分的面积。

解.

$$
\begin{aligned}
&\boldsymbol{r}=\Big(x,y,\frac{xy}{a}\Big),x^2+y^2\le a\\
&\boldsymbol{r}_x=\Big(1,0,\frac{y}{a}\Big),\boldsymbol{r}_y=\Big(0,1,\frac{x}{a}\Big)\\
&\boldsymbol{r}_x\times\boldsymbol{r}_y=\Big(-\frac{y}{a},\frac{x}{a},1\Big),||\boldsymbol{r}_x\times\boldsymbol{r}_y||=\sqrt{1+\frac{x^2+y^2}{a^2}}\\
&\sigma(D)=\iint\limits_{x^2+y^2\le a}\sqrt{1+\frac{x^2+y^2}{a^2}}\mathrm{d}x\mathrm{d}y=\frac{2\pi a^2(2\sqrt{2}-1)}{3}
\end{aligned}
$$

□

12.1.7 螺旋面 $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\\ z=h\theta\end{cases},(0<r<a,0\le\theta\le 2\pi)$ 的面积。

解.

$$
\begin{aligned}
&\boldsymbol{R}(r,\theta)=(r\cos\theta,r\sin\theta,h)\\
&\boldsymbol{R}_r=(\cos\theta,\sin\theta,0),\boldsymbol{R}_\theta=(-r\sin\theta,r\cos\theta,h)\\
&\boldsymbol{R}_r\times\boldsymbol{R}_\theta=(h\sin\theta,-h\cos\theta,r),||\boldsymbol{R}_r\times\boldsymbol{R}_\theta||=\sqrt{h^2+r^2}\\
&\sigma(D)=\iint\limits_{\substack{0<r<a\\0\le\theta\le2\pi}}\sqrt{h^2+r^2}\mathrm{d}r\mathrm{d}\theta=2\pi\int_0^{\frac{a}{h}}h^2\sqrt{1+t^2}\mathrm{d}t
\end{aligned}
$$

计算 $\int_0^x\sqrt{1+t^2}\mathrm{d}t$: 作换元 $t=\sinh u$, 则 $u=\log(t+\sqrt{1+t^2})$.

$$
\int_0^x\sqrt{1+t^2}\mathrm{d}t=\int_0^{u_0}\cosh u\mathrm{d}\sinh u=\int_0^{u_0}\cosh^2 u\mathrm{d}u=\int_0^{u_0}\cosh^2 u\mathrm{d}u=\frac{1}{4}\sinh 2u_0+\frac{u_0}{2}=\frac{1}{2}x\sqrt{1+x^2}+\frac{1}{2}\log(x+\sqrt{1
$$

故带入上式得

$$
\sigma(D)=2\pi h^2\left[\frac{1}{2}\frac{a}{h}\sqrt{1+\frac{a^2}{h^2}}+\frac{1}{2}\log\left(\frac{a}{h}+\sqrt{1+\frac{a^2}{h^2}}\right)\right]=\pi a\sqrt{h^2+a^2}+\pi h^2\log\left(\frac{a}{h}+\sqrt{1+\frac{a^2}{h^2}}\right)
$$

□

15 Week 15

12.2.1 $\int_\Sigma\frac{\mathrm{d}\sigma}{(1+x+y)^2}$, Σ 是四面体 $x+y+z\le 1(x,y,z\ge 0)$ 的边界。

解. 令 $\Sigma_1=\Sigma\cap\{x+y+z=1\},\Sigma_2=\Sigma\cap\{x=0\},\Sigma_3=\Sigma\cap\{y=0\},\Sigma_4=\Sigma\cap\{z=0\}$. 分别计算。

Σ_1:

$$
\begin{aligned}
&\boldsymbol{r}=(x,y,1-x-y),\boldsymbol{r}_x=(1,0,-1),\boldsymbol{r}_y=(0,1,-1)\\
&\boldsymbol{r}_x\times\boldsymbol{r}_y=(1,1,1),||\boldsymbol{r}_x\times\boldsymbol{r}_y||=\sqrt{3}\\
&\int_{\Sigma_1}\frac{\mathrm{d}\sigma}{(1+x+y)^2}=\iint\limits_{\substack{0\le x+y\le1\\0\le x\le1\\0\le y\le1}}\frac{\sqrt{3}}{(1+x+y)^2}\mathrm{d}x\mathrm{d}y=\int_0^1\int_0^{1-x}\frac{\sqrt{3}}{(1+x+y)^2}\mathrm{d}y\mathrm{d}x=\sqrt{3}\Big(\log 2-\frac{1}{2}\Big).
\end{aligned}
$$

Σ_2:

$$\boldsymbol{r} = (0, y, z), ||\boldsymbol{r}_y \times \boldsymbol{r}_z|| = 1$$

$$\int_{\Sigma_2} \frac{\mathrm{d}\sigma}{(1+x+y)^2} = \iint\limits_{\substack{0\le y+z\le 1\\ 0\le y\le 1\\ 0\le z\le 1}} \frac{1}{(1+y)^2}\mathrm{d}y\mathrm{d}z = \int_0^1 \int_0^{1-z} \frac{1}{(1+y)^2}\mathrm{d}y\mathrm{d}z = 1 - \log 2.$$

Σ_3: 由于 x 和 y 对称，故 $\displaystyle\int_{\Sigma_3} \frac{\mathrm{d}\sigma}{(1+x+y)^2} = \int_{\Sigma_2} \frac{\mathrm{d}\sigma}{(1+x+y)^2}$.

Σ_4:

$$\boldsymbol{r} = (x, y, 0), ||\boldsymbol{r}_x \times \boldsymbol{r}_y|| = 1$$

$$\int_{\Sigma_4} \frac{\mathrm{d}\sigma}{(1+x+y)^2} = \iint\limits_{\substack{0\le x+y\le 1\\ 0\le x\le 1\\ 0\le y\le 1}} \frac{1}{(1+x+y)^2}\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^{1-x} \frac{1}{(1+x+y)^2}\mathrm{d}y\mathrm{d}x = \log 2 - \frac{1}{2}$$

故

$$\begin{aligned}\int_{\Sigma} \frac{\mathrm{d}\sigma}{(1+x+y)^2} &= \int_{\Sigma_1} \frac{\mathrm{d}\sigma}{(1+x+y)^2} + \int_{\Sigma_2} \frac{\mathrm{d}\sigma}{(1+x+y)^2} + \int_{\Sigma_3} \frac{\mathrm{d}\sigma}{(1+x+y)^2} + \int_{\Sigma_4} \frac{\mathrm{d}\sigma}{(1+x+y)^2}\\ &= (\sqrt{3}-1)\log 2 + \frac{3-\sqrt{3}}{2}.\end{aligned}$$

□

12.2.2 $\displaystyle\int_{\Sigma} |xyz|\mathrm{d}\sigma, \Sigma : z = x^2 + y^2, z \le 1.$

解.

$$\begin{aligned}&\boldsymbol{r} = (x, y, x^2+y^2), \boldsymbol{r}_x = (1, 0, 2x), \boldsymbol{r}_y = (0, 1, 2y)\\ &\boldsymbol{r}_x \times \boldsymbol{r}_y = (-2x, -2y, 1), ||\boldsymbol{r}_x \times \boldsymbol{r}_y|| = \sqrt{1+4(x^2+y^2)}\\ &\int_{\Sigma} |xyz|\mathrm{d}\sigma = \iint\limits_{x^2+y^2\le 1} |xy|(x^2+y^2)\sqrt{1+4(x^2+y^2)}\mathrm{d}x\mathrm{d}y\\ &\qquad = \int_0^{2\pi} \int_0^1 r^5\sqrt{4r^2+1}|\cos\theta\sin\theta|\mathrm{d}r\mathrm{d}\theta\\ &\qquad =2\int_0^1 r^5\sqrt{4r^2+1}\mathrm{d}r\\ &\qquad \overset{u=\sqrt{4r^2+1}}{=} \int_1^{\sqrt{5}} u^2\Big(\frac{u^2-1}{4}\Big)^2\mathrm{d}u\\ &\qquad =\frac{125\sqrt{5}-1}{420}.\end{aligned}$$

□

12.2.3 $\displaystyle\int_{\Sigma} (xy+yz+zx)\mathrm{d}\sigma, \Sigma : z = \sqrt{x^2+y^2}$ 被圆柱面 $x^2+y^2 = 2x$ 截下的部分。

解.

$$\begin{aligned}&\boldsymbol{r} = (x, y, \sqrt{x^2+y^2}), \boldsymbol{r}_x = (1, 0, \frac{x}{z}), \boldsymbol{r}_y = (0, 1, \frac{y}{z})\\ &\boldsymbol{r}_x \times \boldsymbol{r}_y = (-\frac{x}{z}, -\frac{y}{z}, 1), ||\boldsymbol{r}_x \times \boldsymbol{r}_y|| = \sqrt{1+\frac{x^2+y^2}{z^2}} = \sqrt{2}\end{aligned}$$

$$\int_\Sigma (xy+yz+zx)\mathrm{d}\sigma = \iint\limits_{x^2+y^2\le 2x} \sqrt{2}\big(xy+(x+y)\sqrt{x^2+y^2}\big)\mathrm{d}x\mathrm{d}y$$
$$=\sqrt{2}\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3\cos\theta\sin\theta + r^3(\cos\theta+\sin\theta)\mathrm{d}r\mathrm{d}\theta$$
$$=\frac{64\sqrt{2}}{15}$$

□

12.3.1 计算第二型曲面积分。

(1) $\iint_\Sigma x^4\mathrm{d}y\mathrm{d}z + y^4\mathrm{d}z\mathrm{d}x + z^4\mathrm{d}x\mathrm{d}y, \Sigma: x^2+y^2+z^2=a^2$, 内侧;

(2) $\iint_\Sigma xz\mathrm{d}y\mathrm{d}z + yz\mathrm{d}z\mathrm{d}x + x^2\mathrm{d}x\mathrm{d}y, \Sigma: x^2+y^2+z^2=a^2$, 外侧;

(3) $\iint_\Sigma f(x)\mathrm{d}y\mathrm{d}z + g(y)\mathrm{d}z\mathrm{d}x + h(z)\mathrm{d}x\mathrm{d}y, \Sigma: [0,a]\times[0,b]\times[0,c]$ 的边界，外侧;

(4) $\iint_\Sigma z\mathrm{d}x\mathrm{d}y, \Sigma: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, 外侧;

(5) $\iint_\Sigma (y-z)\mathrm{d}y\mathrm{d}z + (z-x)\mathrm{d}z\mathrm{d}x + (x-y)\mathrm{d}x\mathrm{d}y, \Sigma: z=\sqrt{x^2+y^2}, z\le h$, 外侧。

解. (1) $\iint_\Sigma x^4\mathrm{d}y\mathrm{d}z = \iint_{\Sigma\cap\{x>0\}} x^4\mathrm{d}y\mathrm{d}z + \iint_{\Sigma\cap\{x<0\}} x^4\mathrm{d}y\mathrm{d}z, \boldsymbol{n} = -\frac{(x,y,z)}{a}$, 故在 $\Sigma\cap\{x>0\}$ 和 $\Sigma\cap\{x<0\}$ 上对应位置的法向量是反向的，由对称性知 $\iint_\Sigma x^4\mathrm{d}y\mathrm{d}z = 0$, 又因为 x,y,z 三者对称，故

$$\iint_\Sigma x^4\mathrm{d}y\mathrm{d}z + y^4\mathrm{d}z\mathrm{d}x + z^4\mathrm{d}x\mathrm{d}y = 0.$$

(2) $\boldsymbol{F} = (xz, yz, x^2), \boldsymbol{n} = \frac{(x,y,z)}{a}$.

$$\iint_\Sigma xz\mathrm{d}y\mathrm{d}z + yz\mathrm{d}z\mathrm{d}x + x^2\mathrm{d}x\mathrm{d}y = \iint_\Sigma \frac{1}{a}(x^2z+y^2z+x^2z)\mathrm{d}\sigma = \frac{1}{a}\iint_\Sigma (2x^2+y^2)z\mathrm{d}\sigma \overset{\text{关于}z\text{奇}}{=\!=\!=} 0.$$

(3) 根据各面的法向量得到

$$\iint_\Sigma f(x)\mathrm{d}y\mathrm{d}z + g(y)\mathrm{d}z\mathrm{d}x + h(z)\mathrm{d}x\mathrm{d}y$$
$$=\iint\limits_{\Sigma\cap\{x=a\}} f(x)\mathrm{d}\sigma + \iint\limits_{\Sigma\cap\{x=0\}} -f(x)\mathrm{d}\sigma + \iint\limits_{\Sigma\cap\{y=b\}} g(y)\mathrm{d}\sigma + \iint\limits_{\Sigma\cap\{y=0\}} -g(y)\mathrm{d}\sigma + \iint\limits_{\Sigma\cap\{z=c\}} h(z)\mathrm{d}\sigma + \iint\limits_{\Sigma\cap\{z=0\}} -h(z)\mathrm{d}\sigma$$
$$=\big(f(a)-f(0)\big)bc + \big(g(b)-g(0)\big)ac + \big(h(c)-h(0)\big)ab.$$

(4)

$$\iint_\Sigma z\mathrm{d}x\mathrm{d}y = \iint\limits_{\Sigma\cap\{z>0\}} z\mathrm{d}x\mathrm{d}y + \iint\limits_{\Sigma\cap\{z<0\}} z\mathrm{d}x\mathrm{d}y$$
$$= 2c\iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\mathrm{d}x\mathrm{d}y$$
$$= \frac{4\pi abc}{3}$$

(5)

$$\iint_\Sigma (y-z)\mathrm{d}y\mathrm{d}z = \iint_{\Sigma\cap\{x>0\}} (y-z)\mathrm{d}y\mathrm{d}z + \iint_{\Sigma\cap\{x<0\}} (y-z)\mathrm{d}y\mathrm{d}z$$

由图形知 $x>0$ 和 $x<0$ 时法向相反，故 $\iint_{\Sigma\cap\{x>0\}} (y-z)\mathrm{d}y\mathrm{d}z = -\iint_{\Sigma\cap\{x<0\}} (y-z)\mathrm{d}y\mathrm{d}z$, 从而

$$\iint_\Sigma (y-z)\mathrm{d}y\mathrm{d}z = \iint_{\Sigma\cap\{x>0\}} (y-z)\mathrm{d}y\mathrm{d}z + \iint_{\Sigma\cap\{x<0\}} (y-z)\mathrm{d}y\mathrm{d}z = 0.$$

再由 x 和 y 的对称性知 $\iint_\Sigma (z-x)\mathrm{d}z\mathrm{d}x = 0$.

$$\iint_\sigma (x-y)\mathrm{d}x\mathrm{d}y = -\iint_\sigma (y-x)\mathrm{d}\sigma = -\iint\limits_{x^2+y^2\le h^2} (y-x)\mathrm{d}x\mathrm{d}y \overset{*}{=} 0,$$

其中 * 处是因为积分关于 x,y 是对称的。 □

12.3.2 给定流速场 $\boldsymbol{F}=(y,z,x)$, 封闭曲面 $x^2+y^2=R^2, z=0, z=h$. 计算 $\boldsymbol{F}$ 流向曲面之外的流量。

解. 设 $\boldsymbol{n}$ 是曲面的外法向，则流量为 $\iint_\Sigma \boldsymbol{F}\cdot\boldsymbol{n}\mathrm{d}\sigma$.

记 $\Sigma_1: x^2+y^2=R^2, 0\le z\le h, \Sigma_2: x^2+y^2\le R^2, z=0, \Sigma_3: x^2+y^2\le R^2, z=h$.

在这三部分上的法向量分别是 $\boldsymbol{n}_1=\frac{(x,y,0)}{R}, \boldsymbol{n}_2=(0,0,-1), \boldsymbol{n}_3=(0,0,1)$.

分别计算

$$\begin{aligned}
\iint_{\Sigma_1} \boldsymbol{F}\cdot\boldsymbol{n}\mathrm{d}\sigma &= \iint_{\Sigma_1} \frac{xy+yz}{R}\mathrm{d}\sigma = 0 \quad &(\text{关于 } y \text{ 奇});\\
\iint_{\Sigma_2} \boldsymbol{F}\cdot\boldsymbol{n}\mathrm{d}\sigma &= \iint_{\Sigma_2} -x\mathrm{d}\sigma = 0 \quad &(\text{关于 } x \text{ 奇});\\
\iint_{\Sigma_3} \boldsymbol{F}\cdot\boldsymbol{n}\mathrm{d}\sigma &= \iint_{\Sigma_3} x\mathrm{d}\sigma = 0 \quad &(\text{关于 } x \text{ 奇}).
\end{aligned}$$

故总流量为 0. □

12.3.3 设 $\Sigma: z=f(x,y), (x,y)\in D$, 法向量向上，求证：

(1) $\iint\limits_\Sigma P\mathrm{d}y\mathrm{d}z = -\iint\limits_D P\big(x,y,f(x,y)\big)\frac{\partial f}{\partial x}\mathrm{d}x\mathrm{d}y$;

(2) $\iint\limits_\Sigma Q\mathrm{d}z\mathrm{d}x = -\iint\limits_D Q\big(x,y,f(x,y)\big)\frac{\partial f}{\partial y}\mathrm{d}x\mathrm{d}y$.

解.

$$\boldsymbol{r}=\big(x,y,f(x,y)\big), \boldsymbol{r}_x=(1,0,f_x), \boldsymbol{r}_y=(0,1,f_y)$$

$$\boldsymbol{r}_x\times\boldsymbol{r}_y=(-f_x,-f_y,1), \boldsymbol{n}=\frac{\boldsymbol{r}_x\times\boldsymbol{r}_y}{||\boldsymbol{r}_x\times\boldsymbol{r}_y||}$$

(1)

$$\iint\limits_\Sigma P\mathrm{d}y\mathrm{d}z = \iint\limits_\Sigma P\cdot\frac{-f_x}{||\boldsymbol{r}_x\times\boldsymbol{r}_y||}\mathrm{d}\sigma = \iint\limits_\Sigma P\cdot\frac{-f_x}{||\boldsymbol{r}_x\times\boldsymbol{r}_y||}||\boldsymbol{r}_x\times\boldsymbol{r}_y||\mathrm{d}x\mathrm{d}y = -\iint\limits_D P\big(x,y,f(x,y)\big)\frac{\partial f}{\partial x}\mathrm{d}x\mathrm{d}y.$$

(2) 同理。 □

16 Week 16

12.4.1 利用 Gauss 公式，计算下列积分:

(1) $\iint\limits_{\Sigma} x^2dydz + y^2dzdx + z^2dxdy$,$\Sigma$ 为球面 $x^2 + y^2 + z^2 = R^2$, 方向朝外;

(2) $\iint\limits_{\Sigma} xydydz + yzdzdx + zxdxdy$,$\Sigma$ 是由四张平面 $x = 0, y = 0, z = 0$ 和 $x + y + z = 1$ 围成的封闭曲面，方向朝外;

(3) $\iint\limits_{\Sigma} (x - y)dydz + (y - z)dzdx + (z - x)dxdy$,$\Sigma$ 是曲面 $z = x^2 + y^2(z \le 1)$, 方向朝下;

(4) $\iint\limits_{\Sigma} x^2dydz + y^2dzdx + z^2dxdy$,$\Sigma$ 是曲面 $z^2 = x^2 + y^2(0 \le z \le 1)$, 方向朝下.

解. (1) 由 Gauss 公式和对称性,

$$\begin{aligned}\iint\limits_{\Sigma} x^2dydz + y^2dzdx + z^2dxdy &= \iiint\limits_{\Omega} 2(x + y + z)dxdydz \\ &= 0.\end{aligned}$$

(2) 由 Gauss 公式和对称性,

$$\begin{aligned}\iint\limits_{\Sigma} xydydz + yzdzdx + zxdxdy &= \iiint\limits_{\Omega} x + y + zdxdydz \\ &= 3\iint\limits_{\Omega} xdxdydz \\ &= \frac{1}{8}\end{aligned}.$$

(3) 先补上 $z = 1$ 的平面上一部分使其为封闭的曲面，再由 Gauss 公式得,

$$\begin{aligned}\iint\limits_{\Sigma} (x - y)dydz + (y - z)dzdx + (z - x)dxdy &= \iiint\limits_{\Omega} 3dxdydz - \iint\limits_{\substack{x^2+y^2\le 1\\ z=1}} (z - x)dxdy \\ &= \frac{3\pi}{2} - \pi \\ &= \frac{\pi}{2}.\end{aligned}$$

(3) 先补上 $z = 1$ 的平面上一部分使其为封闭的曲面，再由 Gauss 公式得

$$\begin{aligned}\iint\limits_{\Sigma} x^2dydz + y^2dzdx + z^2dxdy &= \iiint\limits_{\Omega} 2(x + y + z)dxdydz - \iint\limits_{\substack{x^2+y^2\le 1\\ z=1}} z^2dxdy \\ &= 2\iiint\limits_{\Omega} zdxdydz - \pi \\ &= -\frac{\pi}{2}.\end{aligned}$$

□

12.4.2 设 Ω 是一闭域，向量 $\mathbf{n}$ 是 $\partial\Omega$ 的单位外法向量,$\mathbf{e}$ 是固定的一个向量. 求证:

$$\int_{\partial\Omega}\cos(\mathbf{e},\mathbf{n})d\sigma=0.$$

解. 令 $\mathbf{F}=\frac{\mathbf{e}}{||\mathbf{e}||}$, 由 Gauss 公式得到

$$\begin{aligned}\int_{\partial\Omega}\cos(\mathbf{e},\mathbf{n})d\sigma&=\int_{\partial\Omega}\mathbf{F}\cdot\mathbf{n}d\sigma\\&=\int_{\Omega}div(\mathbf{F})d\mu\\&=0.\end{aligned}$$

□

12.4.3 设 Ω 是一闭域,向量 $\mathbf{n}$ 是 $\partial\Omega$ 的单位外法向量,点 $(a,b,c)\notin\partial\Omega$. 令 $\mathbf{p}=(x-a,y-b,z-c)$ 且 $p=||\mathbf{p}||$. 求证:

$$\iiint\limits_{\Omega}\frac{dxdydz}{p}=\frac{1}{2}\int_{\partial\Omega}\cos(\mathbf{p},\mathbf{n})d\sigma.$$

解. 令 $\mathbf{F}=\frac{\mathbf{p}}{||\mathbf{p}||}$, 由 Gauss 公式得到

$$\begin{aligned}\int_{\partial\Omega}\cos(\mathbf{p},\mathbf{n})d\sigma&=\int_{\partial\Omega}\mathbf{F}\cdot\mathbf{n}d\sigma\\&=\int_{\Omega}div(\mathbf{F})d\mu\\&=2\iiint\limits_{\Omega}\frac{dxdydz}{p}.\end{aligned}$$

□

12.4.4 利用 Stokes 公式, 计算下列积分:

(1)$\int_\Gamma ydx+zdy+xdz$,Γ 为圆周 $x^2+y^2+z^2=a^2,x+y+z=0$, 从第一卦限看去，Γ 是逆时针方向绕行的;

(2)$\int_\Gamma(y+z)dx+(z+x)dy+(x+y)dz$,$\Gamma$ 为椭圆 $x^2+y^2=2y,y=z$, 从点 (0,1,0) 向 Γ 看去,Γ 是逆时针方向绕行的;

(3)$\int_\Gamma y^2dx+z^2dy+x^2dz$,$\Gamma$ 为 $x^2+y^2+z^2=a^2,x+y+z=a$, 从原点看去,Γ 是逆时针方向绕行的;

解. (1) 曲面的外法向量为 $\frac{1}{\sqrt{3}}(1,1,1)$. 由 Stokes 公式,

$$\begin{aligned}\int_\Gamma ydx+zdy+xdz&=\iint\limits_{\Omega}\frac{1}{\sqrt{3}}\begin{vmatrix}1&1&1\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\y&z&x\end{vmatrix}d\sigma\\&=\sqrt{3}\iint\limits_{\Omega}d\sigma\\&=-\sqrt{3}\pi a^2.\end{aligned}$$

(2) 曲面的外法向量为 $\frac{1}{\sqrt{2}}(0,1,-1)$. 由 Stokes 公式,

$$\int_\Gamma (y+z)dx+(z+x)dy+(x+y)dz = \iint\limits_\Omega \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 1 & -1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix} d\sigma$$

$$= 0.$$

(3) 曲面的外法向量为 $\frac{1}{\sqrt{3}}(-1,-1,-1)$. 由 Stokes 公式,

$$\begin{aligned}\int_\Gamma y^2dx+z^2dy+x^2dz &= \iint\limits_\Omega \frac{1}{\sqrt{3}}\begin{vmatrix} -1 & -1 & -1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} d\sigma \\ &= \frac{2}{\sqrt{3}}\iint\limits_\Omega (x+y+z)d\sigma \\ &= \frac{2a}{\sqrt{3}}\iint\limits_\Omega d\sigma \\ &= \frac{4\sqrt{3}}{9}\pi a^3.\end{aligned}$$

□

12.4.5 设曲面 Σ 有单位法向量 $\mathbf{n}$,$\mathbf{a}$ 是一个常向量. 求证:

$$\int_{\partial\Sigma} \mathbf{a}\times\mathbf{p}\cdot d\mathbf{p} = 2\iint\limits_\Sigma \mathbf{a}\cdot\mathbf{n}d\sigma.$$

解. 设 $\mathbf{p}=(x,y,z),\mathbf{a}=(a_1,a_2,a_3)$, 则有

$$\mathbf{a}\times\mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2z-a_3y, a_3x-a_1z, a_1y-a_2x)$$

则由 Stokes 公式得

$$\begin{aligned}\int_{\partial\Sigma} \mathbf{a}\times\mathbf{p}\cdot d\mathbf{p} &= \iint\limits_\Sigma \begin{vmatrix} n_x & n_y & n_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2z-a_3y & a_3x-a_1z & a_1y-a_2x \end{vmatrix} d\sigma \\ &= 2\iint\limits_\Sigma a_1n_x+a_2n_y+a_3n_zd\sigma \\ &= 2\iint\limits_\Sigma \mathbf{a}\cdot\mathbf{n}d\sigma.\end{aligned}$$

□

12.4.6 计算 $\int_\Gamma ydx+zdy+xdz$,Γ 是平面 $x+y=2$ 和球面 $x^2+y^2+z^2=2(x+y)$ 交成的圆周, 从原点看去, 顺时针方向是 Γ 的正向.

解. 曲面的外法向量为 $\frac{1}{\sqrt{2}}(1,1,0)$. 由 Stokes 公式,

$$\begin{aligned}\int_\Gamma ydx+zdy+xdz &= \iint\limits_\Omega \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & 1 & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} d\sigma \\ &= -\sqrt{2}\iint\limits_\Omega d\sigma \\ &= -2\sqrt{2}\pi.\end{aligned}$$

□

12.4.7 计算上级的积分, 但 Γ 是曲面 $z=xy$ 和 $x^2+y^2=1$ 的交线, 沿 Γ 的正向行进时, z 轴在左手边.

解. 曲面在 (x,y,z) 处的外法向量为 $\frac{1}{\sqrt{x^2+y^2+1}}(-x,-y,1)$. 由 Stokes 公式,

$$\begin{aligned}\int_\Gamma ydx+zdy+xdz &= \iint\limits_\Omega \begin{vmatrix} n_x & n_y & n_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} d\sigma \\ &= -\iint\limits_\Omega dydz+dzdx+dxdy \\ &= -\iint\limits_\Omega n_x+n_y+n_z d\sigma \\ &= \iint\limits_{x^2+y^2\le 1} (x+y-1)dxdy \\ &= -\pi.\end{aligned}$$

□

12.4.8 设定向曲线 $\Gamma: x^2+y^2+z^2=a^2, x^2+y^2=ax, z\ge 0$, 从点 (a/2,0,0) 看去, 沿逆时针方向行进. 试计算力场 $\mathbf{F}=(y^2,z^2,x^2)$ 沿 Γ 所做的功.

解. 曲面在 (x,y,z) 处的外法向量为 $\frac{1}{a}(-x,-y,-z)$. 由 Stokes 公式,

$$\begin{aligned}\int_\Gamma ydx+zdy+xdz &= \iint\limits_\Omega \begin{vmatrix} n_x & n_y & n_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} d\sigma \\ &= \iint\limits_\Omega -2zdydz-2xdzdx-2ydxdy \\ &= -\iint\limits_\Omega \frac{2}{a}(xz+xy+yz)d\sigma \\ &= -\iint\limits_\Omega \frac{2}{a}xzd\sigma \\ &= \iint\limits_{x^2+y^2\le ax} 2xdxdy \\ &= \frac{\pi}{4}a^3.\end{aligned}$$

注: 也可以用课本第 104 页，这是曲面的定向向下，右边出来正负号。 □

12.5.1 计算:

解. (1)$xzdx\wedge dz+yzdy\wedge dz+yzdx\wedge dy$,
(2)$(x-z)dx\wedge dy\wedge dz$. □

12.5.2 计算 $d\omega$

解. (1)$(y+z)dx+(x+z)dy+(y+x)dz$,
(2)$-ydx\wedge dy$,
(3)$ydz\wedge dx+(x+z)dy\wedge dx$,
(4)$xdx\wedge dy$,
(5)$-(x^2+yze^x)dz\wedge dy-ye^xdz\wedge dy$,
(6)$(y^2-2xz)dx\wedge dy\wedge dz$,
(7)$(x+y+z)dx\wedge dy\wedge dz$. □

13.1.1 设 f,g 为数量场, 证明:

$$\nabla\frac{f}{g}=\frac{1}{g^2}(g\nabla f-f\nabla g).$$

解. 逐个分量直接计算得. □

13.1.2 设 u 为一数量场,$\mathbf{f}$ 为一向量场. 计算 $\nabla(u\circ\mathbf{f})$.

解. 令 $f=(P,Q,R)$ 由链式法则逐个分量计算得到

$$\nabla(u\circ\mathbf{f})=u_1'\nabla P+u_2'\nabla Q+u_3'\nabla R$$

□

13.1.3 设 $\mathbf{p}=(x,y,z)$,$p=||\mathbf{p}||$,f 为单变量函数. 计算:
(2)$\nabla f(p)$;
(4)$\nabla(f(p)\mathbf{p}\cdot\mathbf{a})$, 其中 $\mathbf{a}$ 为常向量

解. (2)

$$\nabla f(p)=\frac{f'(p)}{p}\mathbf{p}$$

(4)

$$\nabla(f(p)\mathbf{p}\cdot\mathbf{a})=\mathbf{p}\cdot\mathbf{a}f'(p)\frac{\mathbf{p}}{p}+f(p)\mathbf{a}$$

□

13.1.4 求数量场 f 沿数量场 g 的梯度方向的变化率, 问何时这个变化率等于零?

解. 由方向导数的计算方法

$$\frac{\partial f}{\partial\nabla g}=\nabla f\cdot\nabla g$$

则在 ∇f 与 ∇g 相互垂直的时候, 变化率为 0. □

13.1.5 设 Ω 是 Gauss 公式中的闭区域, $\mathbf{n}$ 是 $\partial\Omega$ 的单位外法向量场, 数量场 u$\in C^1(\Omega)$, 点 $\mathbf{p}\in\Omega^\circ$. 证明:

$$\nabla u(\mathbf{p})=\lim_{\Omega\to\mathbf{p}}\frac{1}{\mu(\Omega)}\iint\limits_{\partial\Omega}u\mathbf{n}d\sigma.$$

解.

$$\begin{aligned}\lim_{\Omega\to\mathbf{p}}\frac{1}{\mu(\Omega)}\iint\limits_{\partial\Omega}u\mathbf{n}d\sigma&=\lim_{\Omega\to\mathbf{p}}\frac{1}{\mu(\Omega)}(\iint\limits_{\partial\Omega}udydz,\iint\limits_{\partial\Omega}udzdx,\iint\limits_{\partial\Omega}udxdy)\\&=\lim_{\Omega\to\mathbf{p}}\frac{1}{\mu(\Omega)}(\iiint\limits_{\Omega}\frac{\partial u}{\partial x}d\mu,\iiint\limits_{\Omega}\frac{\partial u}{\partial y}d\mu,\iiint\limits_{\Omega}\frac{\partial u}{\partial z}d\mu)\\&=\lim_{\Omega\to\mathbf{p}}(\frac{\partial u}{\partial x}(\xi),\frac{\partial u}{\partial y}(\eta),\frac{\partial u}{\partial z}(\gamma))\\&=\nabla u(\mathbf{p}).\end{aligned}$$

□

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13.2.1 在 $\mathbb{R}^2$ 中, 令 $\mathbf{p}=(x,y)$ 且 $p=||\mathbf{p}||$. 求证: 当 $p>0$ 时，$\log p$ 是调和函数.

解. 在 $\mathbb{R}^2$ 中，计算有

$$\nabla\log p=\frac{\mathbf{p}}{p^2}$$

于是我们有

$$\Delta\log p=\nabla\cdot(\nabla\log p)=\nabla\cdot(\frac{\mathbf{p}}{p^2})=0$$

□

13.2.2 求证:

$$\Delta(fg)=f\Delta(g)+g\Delta(f)+2\nabla f\cdot\nabla g.$$

解.

$$\begin{aligned}\Delta(fg) &= \nabla\cdot(\nabla(fg))\\ &= \nabla\cdot(\nabla(f)g+f\nabla(g))\\ &= \nabla\cdot(\nabla(f)g)+\nabla\cdot(f\nabla(g))\\ &= \nabla\cdot(\nabla(f))g+2\nabla f\cdot\nabla g+f\nabla\cdot(\nabla(g))\\ &= f\Delta(g)+g\Delta(f)+2\nabla f\cdot\nabla g.\end{aligned}$$

□

13.2.3 设 Ω 是 Gauss 公式中的闭区域，u，v$\in C^2(\Omega)$,$\mathbf{n}$ 表示 $\partial\Omega$ 的单位外法向量场, 求证:

(1)$\displaystyle\int_{\partial\Omega}\frac{\partial u}{\partial\mathbf{n}}d\sigma=\int_\Omega\Delta u d\mu$;

(2)$\displaystyle\int_{\partial\Omega}v\frac{\partial u}{\partial\mathbf{n}}d\sigma=\int_\Omega\nabla u\cdot\nabla v d\mu+\int_\Omega v\Delta u d\mu$;

(3)(第二 Green 公式)

$$\int_{\partial\Omega}\begin{vmatrix}\frac{\partial u}{\partial\mathbf{n}} & \frac{\partial v}{\partial\mathbf{n}}\\ u & v\end{vmatrix}d\sigma=\int_\Omega\begin{vmatrix}\Delta u & \Delta v\\ u & v\end{vmatrix}d\sigma.$$

解. (1) 由方向导数的计算方法和 Gauss 公式得到

$$\begin{aligned}\int_{\partial\Omega}\frac{\partial u}{\partial\mathbf{n}}d\sigma &= \int_{\partial\Omega}\nabla u\cdot\mathbf{n}d\sigma\\ &= \int_\Omega\Delta u d\mu.\end{aligned}$$

(2) 由方向导数的计算方法和 Gauss 公式得到

$$\begin{aligned}\int_{\partial\Omega}v\frac{\partial u}{\partial\mathbf{n}}d\sigma &= \int_{\partial\Omega}v\nabla u\cdot\mathbf{n}d\sigma\\ &= \int_\Omega\nabla\cdot(v\nabla u)d\mu\\ &= \int_\Omega\nabla u\cdot\nabla v d\mu+\int_\Omega v\Delta u d\mu.\end{aligned}$$

(3) 在 (2) 中交换 u 和 v 得到

$$\int_{\partial\Omega}u\frac{\partial v}{\partial\mathbf{n}}d\sigma=\int_\Omega\nabla v\cdot\nabla u d\mu+\int_\Omega u\Delta v d\mu,$$

与 (2) 中的式子做差得到

$$\int_{\partial\Omega}\begin{vmatrix}\frac{\partial u}{\partial\mathbf{n}} & \frac{\partial v}{\partial\mathbf{n}}\\ u & v\end{vmatrix}d\sigma=\int_\Omega\begin{vmatrix}\Delta u & \Delta v\\ u & v\end{vmatrix}d\sigma.$$

□

13.2.4 设 u 是 $\mathbb{R}^3$ 中的闭区域 Ω 上的调和函数,$\mathbf{n}$ 表示 $\partial\Omega$ 的单位外法向量. 求证:

(1)$\displaystyle\int_{\partial\Omega}\frac{\partial u}{\partial\mathbf{n}}d\sigma=0$;

(2)$\displaystyle\int_{\partial\Omega}u\frac{\partial u}{\partial\mathbf{n}}d\sigma=\int_\Omega||\nabla u||^2$.

解. 在 13.2.3 中带入 $v = u$ 且 u 为调和函数, 即可得到等式成立。 □

13.3.1 证明:

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}.$$

解. 令 $\mathbb{F} = (P, Q, R)$,

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{F}) &= \Big(\frac{\partial^2 Q}{\partial x \partial y} - \frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial z^2} + \frac{\partial^2 R}{\partial x \partial z}, \frac{\partial^2 R}{\partial y \partial z} - \frac{\partial^2 Q}{\partial z^2} - \frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 P}{\partial y \partial x}, \frac{\partial^2 P}{\partial z \partial x} - \frac{\partial^2 R}{\partial x^2} - \frac{\partial^2 R}{\partial y^2} + \frac{\partial^2 Q}{\partial z \partial y} \Big) \\ &= \nabla \Big(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \Big) - (\Delta P, \Delta Q, \Delta R) \\ &= \nabla(\nabla \cdot \mathbf{F}) - \Delta \mathbf{F}\end{aligned}$$

□

13.3.2 设 Ω 是 Gauss 公式中的闭区域, $\mathbf{n}$ 表示 $\partial\Omega$ 的单位外法向量, 向量场 $\mathbf{F} \in C^1(\Omega)$. 求证:

$$rot\mathbf{F}(\mathbf{p}) = \lim_{\Omega \to \mathbf{p}} \frac{1}{\mu(\Omega)} \int_{\partial\Omega} \mathbf{n} \times \mathbf{F} d\sigma.$$

解. 设 $n = (n_x, n_y, n_z)$,$\mathbf{F} = (P, Q, R)$ 计算可得

$$\begin{aligned}\int_{\partial\Omega} \mathbf{n} \times \mathbf{F} d\sigma &= \int_{\partial\Omega} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ n_x & n_y & n_z \\ P & Q & R \end{vmatrix} d\sigma \\ &= \Big(\int_{\partial\Omega} n_y R - n_z Q d\sigma, \int_{\partial\Omega} n_z P - n_x R d\sigma, \int_{\partial\Omega} n_z P - n_x R d\sigma \Big) \\ &= \Big(\iint_{\Omega} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} d\mu, \iint_{\Omega} \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} d\mu, \iint_{\Omega} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} d\mu \Big).\end{aligned}$$

带入由积分中值定理得

$$\begin{aligned}\lim_{\Omega \to \mathbf{p}} \frac{1}{\mu(\Omega)} \int_{\partial\Omega} \mathbf{n} \times \mathbf{F} d\sigma &= \Big(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \Big) \\ &= rot\mathbf{F}(\mathbf{p}).\end{aligned}$$

□

13.3.3 设 Ω 是 Gauss 公式中的闭区域，数量场 $f \in C^2(\Omega)$, 在 Ω 中处处不为零, 且满足条件

$$div(fgradf) = af, \qquad ||\nabla f||^2 = bf,$$

其中 a 与 b 为常数. 试计算 $\int_{\partial\Omega} \frac{\partial f}{\partial \mathbf{n}} d\sigma$.

解.

$$\begin{aligned}af &= \nabla \cdot (f \nabla f) \\ &= \nabla f \cdot \nabla f + f \Delta f \\ &= bf + f \Delta f,\end{aligned}$$

由 f 处处不为 0, 得到

$$\Delta f = a - b,$$

则计算结果得到

$$\begin{aligned}\int_{\partial\Omega}\frac{\partial f}{\partial \mathbf{n}}d\sigma &= \int_{\Omega}\Delta f d\mu \\ &= (a-b)\mu(\Omega).\end{aligned}$$

□

13.4.1 求下面 $\mathbf{F}$ 的势函数:
(1)$\mathbf{F}=\left(1-\frac{1}{y}+\frac{y}{z},\frac{x}{z}+\frac{x}{y^2},-\frac{xy}{z}\right)$;
(2)$\mathbf{F}=\frac{1}{x^2+y^2+z^2+2xy}(x+y,x+y,z)$.

解. (1) 定义域 $D=(x,y,z)|y\neq 0,z\neq 0$, 分为四个单连通的区域. 先考虑 y 大于 0，z 大于 0 的区域，其他区域同理计算. 由 $\nabla\times\mathbf{F}=0$ 则为无旋场，于是存在势函数。

$$\begin{aligned}\varphi(x,y,z) &= \int_{(0,1,1)}^{(x,y,z)}(1-\frac{1}{y}+\frac{y}{z})dx+(\frac{x}{z}+\frac{x}{y^2})dy-\frac{xy}{z}dz \\ &= \frac{xy}{z}-\frac{x}{y}+x\end{aligned}$$

于是全体势函数为 $\frac{xy}{z}-\frac{x}{y}+x+C$

(2) 定义域 $D=(x,y,z)|x+y\neq 0,z\neq 0$, 分为四个单连通的区域. 先考虑 x+y 大于 0，z 大于 0 的区域，其他区域同理计算由 $\nabla\times\mathbf{F}=0$ 则为无旋场，于是存在势函数。

$$\begin{aligned}\varphi(x,y,z) &= \int_{(0,0,1)}^{(x,y,z)}\frac{x+y}{x^2+y^2+z^2+2xy}dx+\frac{x+y}{x^2+y^2+z^2+2xy}dy\frac{z}{x^2+y^2+z^2+2xy}dz \\ &= \frac{1}{2}\log((x+y)^2+z^2)\end{aligned}$$

于是全体势函数为 $\frac{1}{2}\log((x+y)^2+z^2)+C$

□

13.4.2 计算下列恰当微分的曲线积分:
(1)$\displaystyle\int_{(1,1,1)}^{(2,3,-4)} xdx+y^2dy-z^2dz$;
(2)$\displaystyle\int_{(1,2,3)}^{(0,1,1)} yzdx+xzdy+xydz$;
(3)$\displaystyle\int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)}\frac{xdx+ydy+zdz}{\sqrt{x^2+y^2+z^2}}$, 其中 (x_1,y_1,z_1) 是球面 $x^2+y^2+z^2=a^2$ 上的点,(x_2,y_2,z_2) 是球面 $x^2+y^2+z^2=b^2$ 上的点, 并设 $a>0,b>0$.

解. (1)

$$\begin{aligned}\int_{(1,1,1)}^{(2,3,-4)} xdx+y^2dy-z^2dz &= \frac{1}{2}x^2+\frac{1}{3}y^3-\frac{1}{3}z^3\Big|_{(1,1,1)}^{(2,3,-4)} \\ &= \frac{191}{6}.\end{aligned}$$

(2)

$$\begin{aligned}\int_{(1,2,3)}^{(0,1,1)} yzdx+xzdy+xydz &= xyz\Big|_{(1,2,3)}^{(0,1,1)} \\ &= -6.\end{aligned}$$

(3)

$$\int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} \frac{xdx+ydy+zdz}{\sqrt{x^2+y^2+z^2}} = \sqrt{x^2+y^2+z^2}\Big|_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)}$$
$$= b-a.$$

□

13.4.4 设 f 为单变量的连续函数. 计算:

(1)$\int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} f(x+y+z)(dx+dy+dz)$;

(2)$\int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} f(\sqrt{x^2+y^2+z^2})(xdx+ydy+zdz)$.

解. (1) 令 $F(x)=\int_0^x f(t)dt$

$$\int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} f(x+y+z)(dx+dy+dz) = F(x+y+z)\Big|_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)}$$
$$= F(x_2+y_2+z_2)-F(x_1+y_1+z_1)$$
$$= \int_{x_1+y_1+z_1}^{x_2+y_2+z_2} f(t)dt.$$

(2) 令 $F(x)=\int_0^x f(\sqrt{t})dt$

$$\int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} f(\sqrt{x^2+y^2+z^2})(xdx+ydy+zdz) = \frac{1}{2}F(x^2+y^2+z^2)\Big|_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)}$$
$$= \frac{1}{2}(F(\sqrt{x_2^2+y_2^2+z_2^2})-F(\sqrt{x_1^2+y_1^2+z_1^2}))$$
$$= \frac{1}{2}\int_{\sqrt{x_1^2+y_1^2+z_1^2}}^{\sqrt{x_2^2+y_2^2+z_2^2}} f(\sqrt{t})dt.$$

□

13.4.6 求解下列恰当方程:

(1)$xdx+ydy=0$;

(3)$(x+2y)dx+(2x+y)dy=0$;

(5)$e^y dx+(xe^y-2y)dy=0$;

(7)$\frac{xdy-ydx}{x^2+y^2}=xdx+ydy$.

解. (1) $x^2+y^2=C$

(3)

$$\varphi(x,y) = \int_{(0,0)}^{(x,y)} (x+2y)dx+(2x+y)dy$$
$$= \int_0^x xdx + \int_0^y 2x+ydy$$
$$= \frac{1}{2}x^2+2xy+\frac{1}{2}y^2.$$

于是解为 $x^2+4xy+y^2=C$

(5)

$$\begin{aligned}\varphi(x,y) &= \int_{(0,0)}^{(x,y)} e^y dx + (xe^y - 2y)dy \\ &= \int_0^x dx + \int_0^y (xe^y - 2y)dy \\ &= xe^y - y^2.\end{aligned}$$

于是解为 $xe^y - y^2 = C$

(7)

$$d(\arctan\frac{y}{x}) = d(\frac{1}{2}(x^2+y^2))$$

于是解为 $\arctan\frac{y}{x} - \frac{1}{2}(x^2+y^2) = C$

□

13.5.1 证明下列向量场都是 $\mathbb{R}^3$ 中的旋度场，并求其向量势:

(1)$\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$;

(2)$\mathbf{F} = xy\mathbf{i} + -y^2\mathbf{j} + yz\mathbf{k}$;

(3)$\mathbf{F} = (z-y)\mathbf{i} + (x-z)\mathbf{j} + (y-x)\mathbf{k}$.

解. (1) 令 $\mathbf{G} = (P, Q, 0)$ 满足 $\nabla \times \mathbf{G} = \mathbf{F}$ 于是得到

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = z\mathbf{i} + x\mathbf{j} + z\mathbf{k},$$

于是得到

$$\begin{cases} -\frac{\partial Q}{\partial z} = z \\ \frac{\partial P}{\partial z} = x \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y \end{cases}$$

可取出一组解

$$\mathbf{G} = (xz, -\frac{1}{2}z^2 + xy, 0)$$

则所有的向量势为

$$(xz, -\frac{1}{2}z^2 + xy, 0) + \nabla\varphi$$

(2) 令 $\mathbf{G} = (P, Q, 0)$ 满足 $\nabla \times \mathbf{G} = \mathbf{F}$ 于是得到

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = xy\mathbf{i} - y^2\mathbf{j} + (y-x)\mathbf{k},$$

于是得到

$$\begin{cases} -\frac{\partial Q}{\partial z} = xy \\ \frac{\partial P}{\partial z} = -y^2 \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = yz \end{cases}$$

可取出一组解

$$\mathbf{G} = (-y^2 z, -xyz, 0)$$

则所有的向量势为

$$(-y^2 z, -xyz, 0) + \nabla\varphi$$

(3) 令 $\mathbf{G} = (P, Q, 0)$ 满足 $\nabla \times \mathbf{G} = \mathbf{F}$ 于是得到

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = (z-y)\mathbf{i} + (x-z)\mathbf{j} + (y-x)\mathbf{k},$$

于是得到

$$\begin{cases} -\frac{\partial Q}{\partial z} = z - y \\ \frac{\partial P}{\partial z} = x - z \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y - x \end{cases}$$

可取出一组解

$$\mathbf{G} = (xz - \frac{1}{2}z^2 - \frac{1}{2}y^2 + xy, -\frac{1}{2}z^2 + yz, 0)$$

则所有的向量势为

$$(xz - \frac{1}{2}z^2 - \frac{1}{2}y^2 + xy, -\frac{1}{2}z^2 + yz, 0) + \nabla\varphi$$

□

13.5.2 设 Ω 是 $\mathbb{R}^3$ 中关于 $\mathbf{A} = (x_0, y_0, z_0) \neq 0$ 的星形域. 如果 $\mathbf{F}$ 是 Ω 中的无旋场, 即 $div\mathbf{F} = 0$, 证明:$\mathbf{F}$ 必为 Ω 中的旋度场.

解. 见课本. □

13.6.1 在柱坐标中, 设流体的速度 $\mathbf{v}$ 在正交曲线坐标系下的分量 v_r, v_θ, v_z. 求证: 这时的连续性方程是

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0.$$

解. 连续性方程为

$$\frac{\partial \rho}{\partial t} + div(\rho \mathbf{v}) = 0$$

在柱坐标下 $\mathbf{f} = (x, y, z)$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

于是我们得到

$$\begin{cases} \frac{\partial \mathbf{f}}{\partial r} = (\cos\theta, \sin\theta, 0) \\ \frac{\partial \mathbf{f}}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0) \\ \frac{\partial \mathbf{f}}{\partial z} = (0, 0, 1) \end{cases}$$

计算得到

$$h_r = 1 \qquad h_\theta = r \qquad h_z = 1$$

由题设得

$$\mathbf{v} = v_r \frac{\partial \mathbf{f}}{\partial r} + v_\theta \frac{\partial \mathbf{f}}{\partial \theta} + v_z \frac{\partial \mathbf{f}}{\partial z}$$

带入正交标架下的散度表示

$$\nabla \cdot (\rho \mathbf{v}) = \frac{1}{r}(\frac{\partial \rho v_r r}{\partial r} + \frac{\partial \rho v_\theta}{\partial \theta} + \frac{\partial \rho v_z r}{\partial z})$$

带入连续性方程得到柱坐标下的方程

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (prv_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0.$$

□