Solution 1 for 2019∼ 2020 USTC class "Physics of Quantum Information"

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1. Describe and prove the no-cloning theorem.

   Answer: The no-cloning theorem states that it is impossible to create an identical copy of an arbitrary unknown quantum state. The prove can be seen in the "Box 12.1" on the page 532 of "Quantum compution and quantum information" by Nielsen.

2. Prove that non-orthogonal states can't be reliably distinguished.

   Answer: The proof can be seen in the "Box 2.3" on the page 87 of "Quantum compution and quantum information" by Nielsen.

3. Find the eigenvectors, eigenvalues, and diagonal representations of the Pauli matrices X, Y and Z.

   Answer:

   The calculation is omitted.

4. Write down the commutation relations and anti-commutation relations for the Pauli matrices and prove them.

   Answer:

   The commutation relations:

   $$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k,$$

   the anti-commutation relations:

   $$\{\sigma_i, \sigma_j\} = 2\delta ij\mathbf{I},$$

   where $i, j, k = 1, 2, 3$.

   The proof is omitted.

5. Prove the $Cauchy - Schwarz$ inequality that for any two vectors $|v\rangle$ and $|w\rangle$, $|\langle v|w\rangle|^2 \leq \langle v|v\rangle \langle w|w\rangle$.

   Answer: The proof can be seen in the "Box 2.1" on the page 68 of "Quantum compution and quantum information" by Nielsen.

6. Let $\vec{v}$ be any real, three-dimensional unit vector and θ a real number. Prove that

$$\exp(i\theta\vec{v}\cdot\vec{\sigma}) = \cos(\theta)I + i\sin(\theta)\vec{v}\cdot\vec{\sigma},$$

   where $\vec{v}\cdot\vec{\sigma} = \sum_{i=1}^{3} v_i\sigma_i$.

   Answer: Using the properties of Pauli matrices

$$\begin{cases} \sigma_i^2 = I, \\ \sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k, \end{cases}$$

   it can be proved that $(\vec{v}\cdot\vec{\sigma})^2 = I$.

   From Taylor series expansion

$$f(x) = f(0) + f'(0)x + ... = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

   we can get

$$\exp(i\theta A) = \cos(\theta)I + i\sin(\theta)A,$$
$$\exp(-i\theta A) = \cos(\theta)I + i\sin(\theta)A,$$

   where A is an operator, $A^2 = I$.

   Then, we obtain

$$\exp(i\theta\vec{v}\cdot\vec{\sigma}) = \cos(\theta)I + i\sin(\theta)\vec{v}\cdot\vec{\sigma},$$

   if set $A = \vec{v}\cdot\vec{\sigma}$.

7. Prove that for any 2-dimension linear operator A,

$$A = \frac{1}{2}Tr(A)Id + \frac{1}{2}\sum_{k=1}^{3} Tr(A\sigma_k)\sigma_k,$$

   in which $\sigma_k (k = 1, 2, 3)$ are Pauli matrices.

Answer:

Suppose that $A = \sum_{j=0}^{3} a_j\sigma_j$, where $\sigma_0 = Id$ and $\sigma_k (k = 1, 2, 3)$ are Pauli matrices, then

$$A\sigma_k = \sum_{j=0}^{3} a_j\sigma_j\sigma_k,$$

since

$$Tr(\sigma_j\sigma_k) = 2\delta_{jk},$$

then

$$Tr(A\sigma_k) = \sum_{j=0}^{3} Tr(a_j\sigma_j\sigma_k) = \sum_{j=0}^{3} 2a_j\delta_{jk} = 2a_k,$$

so

$$a_k = \frac{1}{2}Tr(A\sigma_k),$$

then we get that

$$A = \frac{1}{2}Tr(A)Id + \frac{1}{2}\sum_{k=1}^{3} Tr(A\sigma_k)\sigma_k,$$

8. If A and B are two linear operators, show that

$$Tr(AB) = Tr(BA).$$

Answer:

$$Tr(AB) = \sum_i\sum_j a_{ij}b_{ji} = \sum_i\sum_j b_{ij}a_{ji} = Tr(BA)$$

9. Let ρ be a density operator.

(1). Show that ρ can be written as

$$\rho = \frac{\boldsymbol{I} + \boldsymbol{r}\cdot\boldsymbol{\sigma}}{2}$$

where r is a real three-dimensional vector such that $||\boldsymbol{r}|| \leq 1$.

(2). Show that $Tr(\rho^2) \leq 1$, with equality if and only if ρ is a pure state.

(3). Show that a state ρ is a pure state if and only if $||\boldsymbol{r}|| = 1$.

Answer:

(1). Using the conclusion derived in the previous problem, we get

$$\rho = \frac{1}{2}Tr(\rho)Id + \frac{1}{2}\sum_{i=1}^{3} Tr(\rho\sigma_i)\sigma_i.$$

By defining $r_i = Tr(\rho\sigma_i), (i = 1, 2, 3)$ and using $Tr(\rho) = 1$, we get

$$\rho = \frac{\boldsymbol{I} + \boldsymbol{r} \cdot \boldsymbol{\sigma}}{2}.$$

(2). Define $\rho = \sum_i p_i \left|\phi_i\right\rangle \left\langle\phi_i\right|$, then

$$\begin{aligned}
\rho^2 &= \sum_i p_i \left|\phi_i\right\rangle \left\langle\phi_i\right| \sum_j p_j \left|\phi_j\right\rangle \left\langle\phi_j\right| \\
&= \sum_{i,j} p_i p_j \left|\phi_i\right\rangle \left\langle\phi_i|\phi_j\right\rangle \left\langle\phi_j\right| \\
&= \sum_i p_i^2 \left|\phi_i\right\rangle \left\langle\phi_i\right|
\end{aligned}$$

then

$$\begin{aligned}
Tr(\rho^2) &= Tr(\sum_i p_i^2 \left|\phi_i\right\rangle \left\langle\phi_i\right|) \\
&= Tr(\sum_i p_i^2 \left\langle\phi_i|\phi_i\right\rangle) \\
&= \sum_i p_i^2
\end{aligned}$$

Since $\sum_i p_i = 1$, then $\sum_i p_i^2 \leq \sum_i p_i = 1$,
then

$$Tr(\rho^2) \leq 1$$

with equality if and only if

$$p_j = 1, p_{i \neq j} = 0$$

when states ρ is a pure state,

(3). Since

$$\rho = \frac{\boldsymbol{I} + \boldsymbol{r} \cdot \boldsymbol{\sigma}}{2},$$

then

$$Tr(\rho^2) = \frac{1}{4}Tr(I + 2\boldsymbol{r} \cdot \boldsymbol{\sigma} + 2||\boldsymbol{r}||^2).$$

using

$$Tr(I) = 2, Tr(\sigma_i) = 0, (i = 1, 2, 3)$$

we get that

$$Tr(\rho^2) = \frac{1}{2}(1 + ||\boldsymbol{r}||^2)$$

since

$$Tr(\rho^2) \leq 1$$

with equality if and only if ρ is a pure state, then

$$||\boldsymbol{r}||^2 \leq 1.$$

$||\boldsymbol{r}|| = 1$ if and only if ρ is a pure state.

10. $\rho_A = \frac{\boldsymbol{I}+\boldsymbol{n_A}\cdot\sigma}{2}$,$\rho_B = \frac{\boldsymbol{I}+\boldsymbol{n_B}\cdot\sigma}{2}$, prove that $Tr(\rho_A\rho_B) = \frac{1+\boldsymbol{n_A}\cdot\boldsymbol{n_B}}{2}$.

    Answer:

    $$\rho_A\rho_B = \frac{1}{4}(\boldsymbol{I} + \boldsymbol{n_A}\cdot\sigma + \boldsymbol{n_B}\cdot\sigma + \boldsymbol{n_A}\cdot\boldsymbol{n_B}\boldsymbol{I} + i\epsilon_{ijk}n_{Ai}n_{Bj}\sigma_k)$$

    As $Tr(\sigma_i) = 0, Tr(\boldsymbol{I}) = 2$,

    $$Tr(\rho_A\rho_B) = \frac{1 + \boldsymbol{n_A}\cdot\boldsymbol{n_B}}{2}.$$

11. Consider an experiment, in which we prepare the state $|0\rangle$ with the probability $|C_0|^2$,and the state $|1\rangle$ with the probability $|C_1|^2$. How to describe this type of quantum state? Compare the differences and similarities between it with the state $C_0|0\rangle + C_1e^{i\theta}|1\rangle$.

    Answer:

    This state is a mixd state, whose density matrix is

    $$\rho = |C_0|^2|0\rangle\langle 0| + |C_1|^2|1\rangle\langle 1| = \begin{pmatrix} |C_0|^2 & 0 \\ 0 & |C_1|^2 \end{pmatrix}$$

    The state $|\psi\rangle = C_0|0\rangle + C_1e^{i\theta}|1\rangle$ is a pure state, whose density matrix is

    $$\begin{aligned}\rho' &= |C_0|^2|0\rangle\langle 0| + |C_1|^2|1\rangle\langle 1| + C_0C_1e^{-i\theta}|0\rangle\langle 1| + C_0C_1e^{i\theta}|1\rangle\langle 0| \\ &= \begin{pmatrix} |C_0|^2 & C_0C_1e^{-i\theta} \\ C_0C_1e^{i\theta} & |C_1|^2 \end{pmatrix}\end{aligned}$$

It's easy to see that heir density matrixes are different. When measuring these two states, if $\{|0\rangle, |1\rangle\}$ basis is used, the probabilities we get $|0\rangle$ and $|1\rangle$ are same; if other basis is used, the probabilities are different.

12. Please prepare the polarized optical quantum state $C_0|0\rangle + C_1 e^{i\theta}|1\rangle$ from an initial state $|0\rangle$, with half wave plate and quarter-wave plate. To implement arbitrary single qubit unitary transformation, how many wave plates are at least needed, and how to perform them?

Answer:

For a rotation δ around an axis in the Bloch Sphere, the operator is given by the Jones matrix

$$U_\delta(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$
$$= \begin{pmatrix} \cos^2(\theta) + e^{-i\delta}\sin^2(\theta) & \cos(\theta)\sin(\theta) - e^{-i\delta}\cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) - e^{-i\delta}\cos(\theta)\sin(\theta) & e^{-i\delta}\cos^2(\theta) + \sin^2(\theta) \end{pmatrix}$$

in which the $0°$ position is defined as the position where horizontal polarized light stays horizontal and vertical polarized stays vertical or in other words the rotation axis concurs with the z-axis in the Bloch Sphere.

A HWP rotates the state vector by $\delta = \pi$. The operator for a HWP reads

$$U_\pi(\theta) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

A QWP rotates the state vector by $\delta = \frac{\pi}{2}$. The operator for a QWP reads

$$U_{\frac{\pi}{2}}(\theta) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + i\cos(2\theta) & i\sin(2\theta) \\ i\sin(2\theta) & 1 - i\cos(2\theta) \end{pmatrix}$$

Since there are three parameters in the unitary matrix $\hat{U}$ except the global phase, two QWPs and one HWP are needed to implement arbitrary single qubit unitary transformation. Solve the relation

$$\hat{U} = e^{i\delta} U_{\frac{\pi}{2}}(\gamma) U_\pi(\beta) U_{\frac{\pi}{2}}(\alpha)$$

i.e.

$$\hat{U} = e^{i\delta}\left(\begin{pmatrix}\cos(\alpha-\gamma)\cos(\alpha-2\beta+\gamma) - i\sin(\alpha+\gamma)\sin(\alpha-2\beta+\gamma) \\ \cos(\alpha-2\beta+\gamma)\sin(\alpha-\gamma) + i\cos(\alpha+\gamma)\sin(\alpha-2\beta+\gamma)\end{pmatrix} \begin{pmatrix} i\cos(\alpha+\gamma)\sin(\alpha-2\beta+\gamma) - \cos(\alpha-2\beta+\gamma)\sin(\alpha-\gamma) \\ \cos(\alpha-\gamma)\cos(\alpha-2\beta+\gamma) + i\sin(\alpha+\gamma)\sin(\alpha-2\beta+\gamma)\end{pmatrix}\right)$$

we can get the angle of the QWPs and HWP. Let the quantum state go through QWP (α), HWP(β) and QWP(γ) one after another we can implement arbitrary single qubit unitary transformation $\hat{U}$.

For example, the angles for the wave plates for the 3 standard directions are

unitary transformation	QWP(α)	HWP(β)	QWP(γ)
$\hat{I}$	0	0	0
$\hat{\sigma_x}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$-\frac{\pi}{4}$
$\hat{\sigma_y}$	$\frac{\pi}{4}$	0	$-\frac{\pi}{4}$
$\hat{\sigma_z}$	$\frac{\pi}{4}$	0	$\frac{\pi}{4}$

To get the state $C_0\left|0\right\rangle + C_1 e^{i\theta}\left|1\right\rangle$, perform the unitary transformation $\hat{U} = e^{i\delta}U_{\frac{\pi}{2}}(\gamma)U_{\pi}(\beta)U_{\frac{\pi}{2}}(\alpha)$ on the state $\left|0\right\rangle$. By solving the relation

$$\begin{pmatrix} C_0 \\ C_1 e^{i\theta}\end{pmatrix} = e^{i\delta}U_{\frac{\pi}{2}}(\gamma)U_{\pi}(\beta)U_{\frac{\pi}{2}}(\alpha)\begin{pmatrix}1 \\ 0\end{pmatrix}$$

i.e.

$$\begin{pmatrix} C_0 \\ C_1 e^{i\theta}\end{pmatrix} = e^{i\delta}\frac{1}{\sqrt{2}}\begin{pmatrix}\cos(\alpha-\gamma)\cos(\alpha-2\beta+\gamma) - i\sin(\alpha+\gamma)\sin(\alpha-2\beta+\gamma) \\ \cos(\alpha-2\beta+\gamma)\sin(\alpha-\gamma) + i\cos(\alpha+\gamma)\sin(\alpha-2\beta+\gamma)\end{pmatrix}$$

we can get the angles of the QWPs and HWP.

13. Suppose a two particle pure state is of the form $\left|\Phi\right\rangle = \sum_{ij} a_{ij}\left|i\right\rangle\left|j\right\rangle$. By defining $A_{ij} = a_{ij}$, calculate the reduced density matrices ρ_A and ρ_B.

Answer:

$$\rho_{AB} = |\phi\rangle\langle\phi| = \sum_{ijkl} a_{ij} a_{kl}^* |ij\rangle\langle kl|$$

$$\begin{aligned}
\rho_A &= Tr_B(\rho_{AB}) \\
&= \sum_{ijkl} a_{ij} a_{kl}^* |i\rangle\langle k|_A \langle l|j\rangle_B \\
&= \sum_{ijkl} a_{ij} a_{kl}^* |i\rangle\langle k| \delta_{lj} \\
&= \sum_{ijk} a_{ij} a_{kj}^* |i\rangle\langle k|
\end{aligned}$$

$$\begin{aligned}
\rho_B &= Tr_A(\rho_{AB}) \\
&= \sum_{ijkl} a_{ij} a_{kl}^* |j\rangle\langle l|_B \langle k|i\rangle_A \\
&= \sum_{ijkl} a_{ij} a_{kl}^* |j\rangle\langle l| \delta_{ki} \\
&= \sum_{ijl} a_{ij} a_{il}^* |j\rangle\langle l|
\end{aligned}$$

Let $A_{ij} = a_{ij}$, then

$$\begin{aligned}
(\rho_A)_{ik} &= \sum_j A_{ij} A_{kj}^* = \sum_j A_{ij} (A^\dagger)_{jk} = (AA^\dagger)_{ik} \\
(\rho_B)_{jl} &= \sum_i A_{ij} A_{il}^* = \sum_i (A^T)_{ji} A_{il}^* = (A^T A^*)_{jl}
\end{aligned}$$

so

$$\begin{aligned}
\rho_A &= AA^\dagger \\
\rho_B &= A^T A^*
\end{aligned}$$

14. Prove that suppose $|\psi\rangle$ is a pure state of a composite system, AB. Then there exist orthonormal states $|i_A\rangle$ for system A, and orthonormal states $|i_B\rangle$ for system B such that

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle,$$

where λ_i are non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$ known as ***Schmidt co-efficients*** .

Answer:

The proof can be seen in the "Theorem 2.7" on the page 109 of "Quantum compution and quantum information" by Nielsen.

15. Prove that suppose $\{|\psi_i\rangle\}, \{|\tilde{\psi_i}\rangle\}$ are two sets of normalized states in space H and they satisfy the conditions that $\langle\psi_i|\psi_j\rangle = \langle\tilde{\psi_i}|\tilde{\psi_j}\rangle$ for $\forall i, j$, then there exist a transformation U, s.t. $U|\psi_i\rangle = |\tilde{\psi_i}\rangle$, and construct this U transformation.

Answer:

Execute Schmidt orthogonalization and normalization of $|\psi_i\rangle$ and $|\tilde{\psi_i}\rangle$ respectively in Hilbert space H.

$$|\psi_i\rangle = \Sigma_j A_{ij}|a_j\rangle, \ \langle a_i|a_j\rangle = \delta_{ij}$$
$$|\tilde{\psi_i}\rangle = \Sigma_j \tilde{A}_{ij}|\tilde{a}_j\rangle, \ \langle\tilde{a}_i|\tilde{a}_j\rangle = \delta_{ij}$$

From $\langle\psi_i|\psi_j\rangle = \langle\tilde{\psi_i}|\tilde{\psi_j}\rangle$, $\forall i, j$, we have $A_{ij} = \tilde{A}_{ij}$. Then define $U = \Sigma_j |\tilde{a}_j\rangle\langle a_j|$, which satisfies that $U|\psi_i\rangle = |\tilde{\psi_i}\rangle$.

16. Suppose ABC is a three component quantum system. Show by example that there are pure quantum states ψ of such systems which can not be written in the form

$$|\psi\rangle = \sum_i \lambda_i \, |i_A\rangle \, |i_B\rangle \, |i_C\rangle$$

where λ_i are real numbers, and $|i_A\rangle \, , |i_B\rangle \, , |i_C\rangle$ are orthonormal bases of the respective systems.

Answer:

For example,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |011\rangle.$$

If $|\psi\rangle$ can be written in the form

$$|\psi\rangle = \sum_i^2 \lambda_i \, |i_A\rangle \, |i_B\rangle \, |i_C\rangle).$$

Then the reduced density matrices ρ_A, ρ_B and ρ_C have the same eigenvalues λ_i^2. However, for the $|\psi\rangle$ given above, ρ_A, ρ_B and ρ_C don't have common eigenvalue. So $|\psi\rangle$ can not be written in the form described as above