2018 ~ 2019

1. $\phi(x, y, f(x, y)) = 0$

$$\frac{\partial \phi}{\partial x} = \phi_1' + \phi_3' \frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = -\phi_3'^{-1} \phi_1' \cancel{\cdot (-1)}$$

$$\phi_3' = \phi' \cdot \left(\frac{1}{x} + \frac{1}{y}\right), \quad \phi_1' = \phi' \cdot \left(1 - \frac{z}{x^2}\right)$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{1 - \frac{z}{x^2}}{\frac{1}{x} + \frac{1}{y}} = -\frac{y - \frac{z}{x} y}{x + y}$$

同理 $\frac{\partial z}{\partial y} = -\phi_3'^{-1} \times \phi_2' = -\frac{\phi' \cdot \left(-\frac{z}{y^2}\right)}{\phi' \cdot \left(\frac{1}{x} + \frac{1}{y}\right)} = \frac{\frac{z}{y} x}{x + y}$

$$\Rightarrow x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \cancel{-\frac{(xy - zy) + zx}{x+y}} - \frac{xy - zy}{x+y} + \frac{zx}{x+y} = \frac{z(x+y) - xy}{x+y}$$

2. $z = \sqrt{x^2 + y^2}$. 面积元 $dS = \sqrt{1 + z_x'^2 + z_y'^2}\, dxdy$

$$= \sqrt{2}\, dxdy$$

设 $D = \{ (x-1)^2 + y^2 \leq 1 \}$. 则

曲面面积 $= \iint_D \sqrt{2}\, dxdy = \sqrt{2}\pi$

3. 椭球参数化：$\begin{cases} x = r\cos\theta\cos\varphi\, a \\ y = r\cos\theta\sin\varphi\, b \\ z = r\sin\theta\, c \end{cases}$

$$I = \cancel{\int_0^1 \int \int_0^1 dr \int_0^{2\pi} d} \int_0^1 \int_0^{2\pi} \int_0^{\pi} \sqrt{1 - r^2} \cdot r^2 abc \sin\theta \, d\theta\, d\varphi\, dr$$

$$= abc \int_0^1 \sqrt{1 - r^2} \cdot r^2\, dr \cdot 4\pi = \frac{\pi^2}{4} abc$$

[因为 $I = \int_0^1 \sqrt{1 - r^2} \cdot r^2\, dr \underset{r = \sin u}{\overset{}{=}} \int_0^{\frac{\pi}{2}} \cos u \cdot \sin u^2 \cdot \cos u\, du$

$$= \int_0^{\frac{\pi}{2}} \sin^2 u\, du - \int_0^{\frac{\pi}{2}} \sin^4 u\, du = \frac{\pi}{4} - \int_0^{\frac{\pi}{2}} \sin^4 u\, du$$

而 $\int_0^{\frac{\pi}{2}} \sin^4 u = -\cos u \sin u^3 \Big|_0^{\frac{\pi}{2}} + 3\int_0^{\frac{\pi}{2}} \cos u^2 \sin^2 u\, du = 3\int_0^{\frac{\pi}{2}} \sin^2 u - 3\int_0^{\frac{\pi}{2}} \sin^4 u$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^4 u\, du = \frac{3}{4} \cdot \frac{\pi}{4}$$

$$\Rightarrow I = \frac{\pi}{4} - \frac{3}{4} \times \frac{\pi}{4} = \frac{\pi}{16}$$

]

4. $\vec{v}=\frac{y^2+z^2}{r^3}i \cancel{+} -\frac{xy}{r^3}j-\frac{xz}{r^5}k$. 定义在 $\mathbb{R}^3\setminus\{0\}$.

(1) 证: $\nabla\times\vec{v}=0$.

计算 $\frac{\partial R}{\partial y}\ \cancel{\frac{\partial Q}{\partial z}}=3xz\frac{1}{r^4}\frac{\partial r}{\partial y}=\frac{3xyz}{r^5}$

$\frac{\partial Q}{\partial z}=3xy\frac{1}{r^4}\frac{\partial r}{\partial z}=\frac{3xyz}{r^5}$

$\Big\} \Rightarrow \frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}=0$

$\frac{\partial P}{\partial z}=-3y^2\frac{1}{r^4}\frac{\partial r}{\partial z}+2\frac{z}{r^3}-3\frac{z^2}{r^4}\frac{\partial r}{\partial z}=\frac{-3y^2z+2zr^2-3z^3}{r^5}=\frac{z(2x^2-y^2-z^2)}{r^5}$

$\frac{\partial R}{\partial x}=-\frac{z}{r^3}+\frac{3xz}{r^4}\frac{x}{r}=\frac{3x^2z-zr^2}{r^5}=\frac{z(2x^2-y^2-z^2)}{r^5}$

等……

(2) ~~取一条路径 $(x,y,z)\sim\infty$ (无穷远点)~~

~~参数化 $r(t)=\begin{pmatrix}x\cdot t\\ y\cdot t\\ z\cdot t\end{pmatrix}$~~

~~则有 $\underset{\overset{\parallel}{0}}{\varphi(\infty)}-\varphi(x,y,z)=\int_1^\infty \frac{y^2+z^2}{r^3t}$~~

取一条路径 $(1,0,0)\sim(x,0,0)\sim(x,y,z)$.

$$\varphi(x,y,z)=\underbrace{\int_{(1,0,0)}^{(x,0,0)}\vec{v}\cdot d\vec{r}}_{\overset{\parallel}{0}}+\int_{(x,0,0)}^{(x,y,z)}\vec{v}\cdot d\vec{r}.$$

参数化: $r(t)=\begin{pmatrix}x\\ ry\\ rz\end{pmatrix}$, $r'(t)=\begin{pmatrix}0\\ y\\ z\end{pmatrix}$

$$\Rightarrow\int_{(x,0,0)}^{(x,y,z)}\vec{v}\cdot d\vec{r}=\int_0^1\left[-\frac{xy^2r}{\sqrt{(x^2+(y^2+z^2)r^2)^3}}-\frac{xz^2r}{\sqrt{(x^2+(y^2+z^2)r^2)^3}}\right]dr$$

$$=+x(y^2+z^2)\cdot\frac{1}{y^2+z^2}(x^2+y^2+z^2)^{-\frac{1}{2}}=\cancel{-rx}\ \frac{x}{r}$$

$\Rightarrow$ 可取势函数为 $\varphi(x,y,z)=\frac{x}{r}$

5. $\nabla\times\vec{v} = \begin{pmatrix} \frac{y}{\sqrt{r^2-z^2}} \\ -\frac{x}{\sqrt{r^2-z^2}} \\ -1-x \end{pmatrix}$ ，球面上法向量 $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

因此 $\iint_S \nabla\times\vec{v}\cdot d\vec{S} = \iint_S (-1-x)z \; dS$

由对称性. 原式 = 0.

6. ~~$I(t)$~~ $I = \int_1^\infty t^2 e^{t(2-t)} \, dt = \int_1^\infty t^2 e^{-(t-1)^2+1} \, dt$

$= e\int_0^\infty (s+1)^2 e^{-s^2} ds$

分别计算 ~~$\int s^2 e$~~ $\int_0^\infty s e^{-s^2} ds =$ ~~$\int_0$~~ $\frac{1}{2}\int_0^\infty e^{-s^2} d(s^2)$

$= \frac{1}{2}\int_0^\infty e^{-t} dt = \frac{1}{2}$

$\int_0^\infty s\cdot s e^{-s^2} ds = -\cancel{s}\frac{1}{2}e^{-s^2} s\Big|_0^\infty + \frac{1}{2}\int_0^\infty \cancel{s} e^{-s^2} ds$

$= \frac{\sqrt{\pi}}{4}$

$\int_0^\infty e^{-s^2} ds = \frac{\sqrt{\pi}}{2}$

$\Rightarrow \underline{I = e\left(1+\frac{3\sqrt{\pi}}{4}\right)}$

7. 对 $\cosh(x-1)$ 作偶延拓. 求得余弦级数为

$a_0 = 2\int_0^1 \cosh(x-1) dx = e - \frac{1}{e}$

$a_n = 2\int_0^1 \cosh(x-1)\cos(n\pi x) dx = \frac{1}{e}\frac{e^x(n\pi\sin n\pi x + \cos n\pi x)}{n^2\pi^2+1}\Big|_0^1 +$

$e\cdot\frac{e^{-x}(n\pi\sin n\pi x - \cos n\pi x)}{n^2\pi^2+1}\Big|_0^1$

$= \frac{\cancel{8}\,1}{n^2\pi^2+1}\left(e(-1)^n - \frac{1}{e}\right) + \frac{1}{1+n^2\pi^2}\left(\cancel{e}(-1)^{n+1} + e\right)$

$= \frac{\cancel{(-1)^n}\,1}{1+n^2\pi^2}(e - e^{-1})$

$\sum_{n\geq1}\frac{1}{n^2\pi^2+1} = \frac{1}{e^2-1}$

$\Uparrow$

$\Rightarrow \underline{f(x) \sim \sum_{n\geq0}\frac{\cancel{(-1)^n}\,1}{1+n^2\pi^2}(e-e^{-1})\cos n\pi x}$

令 $x=0 \Rightarrow$ ~~$1 = (e-e^{-1})\left[1+\sum_{n\geq1}\frac{1}{n^2\pi^2+1}\right]$~~ $\frac{1}{2}(e+e^{-1}) = (e-e^{-1})\left[\frac{1}{2}+\sum_{n\geq1}\cdots\right]$

$$F(t)=\int_0^{\infty}\frac{\sin tx}{1+x^2}\,dx.$$

- $F(t)$ 在 $(0,\infty)$ 上连续： $\forall t_0>0.$ $\int_0^{\infty}\frac{\sin tx}{1+x^2}\,dx$ 在 $[\frac{t_0}{2},\frac{3t_0}{2}]$ 上一致收敛

(Dirichlet 判别法) 故由 Thm 13.28.

- $F(t)$ 在 $(0,\infty)$ 上可导： $\forall t_0>0.$ $\frac{\partial}{\partial t}\left(\frac{\sin tx}{1+x^2}\right)=\frac{x\cos tx}{1+x^2}$

在 $[\frac{t_0}{2},\frac{3t_0}{2}]$ 上一致收敛. 故由 Thm 13.31.

$$F'(t_0)=\int_0^{\infty}\frac{x\cdot\cos t_0x}{1+x^2}\,dx$$

- $F''(t)-F(t)=-\frac{1}{t}.$

$\frac{\partial}{\partial t}\left(\frac{x\cdot\cos tx}{1+x^2}\right)=-\frac{x^2\sin tx}{1+x^2}$ 不一致收敛. 不能直接算！

关键

想要变形. 消去 $F'(t)=\int_0^{\infty}\frac{x\cdot\cos tx}{1+x^2}\,dx$ 分子的 x.

分部积分: $F'(t)=\frac{\sin tx}{t}\cdot\frac{x}{1+x^2}\Big|_0^{\infty}-\frac{1}{t}\int_0^{\infty}\sin tx\left(\frac{x}{1+x^2}\right)'dx$

$$=\frac{1}{t}\int_0^{\infty}\frac{x^2-1}{(1+x^2)^2}\sin tx\,dx=\frac{1}{t}\int_0^{\infty}\frac{\sin tx}{1+x^2}\,dx-\frac{2}{t}\int_0^{\infty}\frac{\sin tx}{(1+x^2)^2}\,d$$

$\Rightarrow F'(t)=\frac{1}{t}F(t)-\frac{2}{t}\int_0^{\infty}\frac{\sin tx}{(1+x^2)^2}\,dx$ 求 $\frac{\partial}{\partial t}$ 后一致收敛.

$$tF'(t)=F(t)-2\int_0^{\infty}\frac{\sin tx}{(1+x^2)^2}\,dx$$

两边求导: $\cancel{F'(t)}+tF''(t)=\cancel{F'(t)}-2\int_0^{\infty}\frac{x\sin tx}{(1+x^2)^2}\,dx$

$$=\frac{1}{1+x^2}\cdot\cos tx\Big|_0^{\infty}+\int_0^{\infty}\frac{x}{1+x^2}\sin tx\,dx$$

$$=-1+F(t)$$

$\Rightarrow tF''(t)=-1+F(t)$