Mathematical Methods of Modern Physics

Solution Manual

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Chapter 1

Problems:

1. Assume that $-3 < \Re z < -2$. Show that for such a z the Gamma function $\Gamma(z)$ is expressed as

$$\Gamma(z) = \int_0^\infty t^{z-1} \left[e^{-t} - 1 + t - \frac{t^2}{2} \right] dt$$

2. Show that $\Gamma(z)$ may be written

$$\Gamma(z) = \int_0^1 dt \, [\ln(1/t)]^{z-1}, \qquad \Re z > 0.$$

3. Show that

$$\int_0^\infty dx \; e^{-x^4} = \Gamma(5/4)$$

4. The wave function of a particle scattered by a Coulomb potential is $\psi(r, \theta)$. At the origin $\psi(0) = e^{-\pi\gamma/2}\Gamma(1 + i\gamma)$, where γ is a real dimensionless constant. Show that:

$$|\psi(0)|^2 = \frac{2\pi\gamma}{e^{2\pi\gamma} - 1}$$

5. The so-called *diagmma function* $\psi(z + 1)$ is defined by

$$\psi(z + 1) = \frac{d}{dz} \ln \Gamma(z + 1)$$

Show that $\psi(z+1)$ has the series expansion

$$\psi(z+1) = -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n) z^{n-1}$$

where $\gamma \approx 0.5772$ and $\zeta(n)$ is the Riemann zeta function $\zeta(n) = \sum_{i=1}^{\infty} i^{-n}$.

6. Let $A_i, (i = 1, 2, \cdots, n)$ be positive real numbers. Start from the obvious identity

$$\frac{1}{A_i} = \int_0^{\infty} ds e^{-sA_i}$$

and prove the Feynman's integral formula.

7. Simplify

$$\int d^d x \; x^\mu x^\nu x^\rho x^\sigma f(x^2)$$

8. Calculate the integral:

$$\int \frac{d^d k}{(2\pi)^d} \frac{(p+k)_\mu k_\nu}{(p+k)^2 k^2}$$

9. Let $B_n(x)$ be the Bernoulli polynomials defined by

$$B_n(x) = \left[\frac{\partial^n}{\partial t^n} \left(\frac{t e^{xt}}{e^t - 1} \right) \right]_{t=0}$$

Show that $B_n(x) = (-1)^n B_n(1-x)$.

10. Regulate the divergent summation

$$\sum_{n=0}^{+\infty} (n + 1/3)$$

11. Start from the Jacobi's triple product identity to show the equivalence between two expressions of the basic theta function.

12. Check the following modular properties of the theta functions:

$$\begin{aligned}
\vartheta_{00}(\nu/\tau, -1/\tau) &= (-i\tau)^{1/2}\exp(\pi i\nu^2/\tau)\vartheta_{00}(\nu,\tau)\\
\vartheta_{01}(\nu/\tau, -1/\tau) &= (-i\tau)^{1/2}\exp(\pi i\nu^2/\tau)\vartheta_{10}(\nu,\tau)\\
\vartheta_{10}(\nu/\tau, -1/\tau) &= (-i\tau)^{1/2}\exp(\pi i\nu^2/\tau)\vartheta_{01}(\nu,\tau)\\
\vartheta_{11}(\nu/\tau, -1/\tau) &= -i(-i\tau)^{1/2}\exp(\pi i\nu^2/\tau)\vartheta_{11}(\nu,\tau)
\end{aligned}$$

Solutions:

1. Relying on the assumption $-3 < \Re z < -2$,

$$\Gamma(z) = \frac{\Gamma(z+3)}{z(z+1)(z+2)} = \frac{1}{z(z+1)(z+2)}\int_0^\infty t^{z+2}e^{-t}dt$$

By introducing a cut-off $\epsilon \geq 0$, we have:

$$\Gamma(z) = I(\epsilon)\Big|_{\epsilon\to 0}$$

where,

$$I(\epsilon) = \frac{1}{z(z+1)(z+2)}\int_\epsilon^\infty t^{z+2}e^{-t}dt$$

Because $-3 < \Re z < -2$, we see that when $\epsilon \to 0$,

$$\begin{aligned}
\int_\epsilon^\infty t^{z+2}e^{-t}dt &= \epsilon^{z+2}e^{-\epsilon} + (z+2)\int_\epsilon^\infty t^{z+1}e^{-t}dt\\
&= \epsilon^{z+2}e^{-\epsilon} + (z+2)\epsilon^{z+1}e^{-\epsilon} + (z+2)(z+1)\int_\epsilon^\infty t^{z}e^{-t}dt\\
&= \epsilon^{z+2}e^{-\epsilon} + (z+2)\epsilon^{z+1}e^{-\epsilon} + (z+2)(z+1)\epsilon^{z}e^{-\epsilon}\\
&\qquad +(z+2)(z+1)z\int_\epsilon^\infty t^{z-1}e^{-t}dt\\
&= \frac{1}{2}z(z+1)\epsilon^{z+2} - z(z+2)\epsilon^{z+1} + (z+2)(z+1)\epsilon^{z}\\
&\qquad +(z+2)(z+1)z\int_\epsilon^\infty t^{z-1}e^{-t}dt
\end{aligned}$$

Therefore,

$$I(\epsilon) = \frac{1}{2}\frac{\epsilon^{z+2}}{z+2} - \frac{\epsilon^{z+1}}{z+1} + \frac{\epsilon^z}{z} + \int_\epsilon^\infty t^{z-1}e^{-t}dt$$

On the other hand,

$$\int_\epsilon^\infty t^{z+n-1}dt = -\frac{\epsilon^{z+n}}{z+n}, \qquad (0 \leq n \leq 3)$$

Therefore,

$$I(\epsilon) = \int_\epsilon^\infty t^{z-1}\left(e^{-t} - 1 + t - \frac{t^2}{2}\right)dt$$

Though the integrand in $I(\epsilon)$ diverges as ϵ^{z+2} if $\epsilon \to 0$, the integration itself converges. This fact implies,

$$\Gamma(z) = \int_0^\infty t^{z-1}\left(e^{-t} - 1 + t - \frac{t^2}{2}\right)dt$$

2. For $\Re z > 0$, Gamma function $\Gamma(z)$ could be expressed as,

$$\Gamma(z) = \int_0^\infty s^{z-1}e^{-s}ds$$

Let $e^{-s} = t$. We see that $t = 1$ if $s = 0$ whereas $t \to 0$ if $s \to \infty$. On the other hand,

$$e^{-s}ds = -d(e^{-s}) = -dt, \qquad s^{z-1} = (-\ln t)^{z-1} = [\ln(1/t)]^{z-1}$$

Therefore,

$$\Gamma(z) = \int_0^1 [\ln(1/t)]^{z-1}dt, \qquad \Re z > 0$$

3. Let $x^4 = t$. We see that $x = t^{1/4}$ and

$$\int_0^\infty dx e^{-x^4} = \frac{1}{4}\int_0^\infty t^{-\frac{3}{4}}e^{-t}dt = \frac{1}{4}\Gamma\left(\frac{1}{4}\right) = \Gamma\left(\frac{5}{4}\right)$$

In the last step, the property $\Gamma(z+1) = z\Gamma(z)$ of Gamma function has been made use of.

4. Because the parameter γ is assumed to be real,

$$\begin{aligned}[\Gamma(1+i\gamma)]^* &= \left(\int_0^\infty t^{i\gamma} e^{-t} dt\right)^* \\ &= \int_0^\infty t^{-i\gamma} e^{-t} dt = \Gamma(1-i\gamma)\end{aligned}$$

Besides, $\Gamma(z)\Gamma(1-z) = \pi \csc(\pi z)$. Therefore,

$$\begin{aligned}|\psi(0)|^2 &= e^{-\pi\gamma}\ \Gamma(1+i\gamma)\Gamma(1-i\gamma) \\ &= e^{-\pi\gamma}\ (i\gamma)\Gamma(i\gamma)\Gamma(1-i\gamma) \\ &= e^{-\pi\gamma} \frac{i\gamma}{\sin(i\gamma\pi)} \\ &= \frac{2\gamma\pi}{e^{2\pi\gamma}-1}\end{aligned}$$

5. Because,

$$\Gamma(z+1) = e^{-\gamma z} \prod_{n=1}^{+\infty} \frac{e^{z/n}}{1+z/n}$$

we have:

$$\ln\Gamma(z+1) = -\gamma z + \sum_{n=1}^{+\infty} \left[\frac{z}{n} - \ln(1+z/n)\right]$$

Consequently,

$$\begin{aligned}\psi(z+1) &= -\gamma + \sum_{n=1}^{+\infty} \frac{1}{n}\left[1-(1+z/n)^{-1}\right] \\ &= -\gamma + \sum_{n=1}^{+\infty} \frac{1}{n}\left[1 - \sum_{i=0}^{+\infty}(-1)^i \frac{z^i}{n^i}\right] \\ &= -\gamma + \sum_{n=1}^{+\infty}\sum_{i=1}^{+\infty}(-1)^{i+1}\frac{z^i}{n^{i+1}} \\ &= -\gamma + \sum_{i=2}^{+\infty}(-1)^i \left(\sum_{n=1}^{+\infty}\frac{1}{n^i}\right) z^{i-1}\end{aligned}$$

That is:

$$\psi(z+1) = -\gamma + \sum_{n=2}^{\infty}(-1)^n \zeta(n) z^{n-1}$$

6. In terms of the identity,

$$\frac{1}{A} = \int_0^{\infty} dt e^{-At}$$

for any $A > 0$, we have :

$$\begin{aligned}
\prod_{i=1}^n A_i^{-1} &= \int_0^{\infty} dt_1 \int_0^{\infty} dt_2 \cdots \int_0^{\infty} dt_n e^{-A_1 t_1 - A_2 t_2 - \cdots - A_n t_n} \\
&= \int_0^{\infty} dt_1 \int_0^{\infty} dt_2 \cdots \int_0^{\infty} dt_n \int_0^{\infty} ds \delta(s - \sum_{i=1}^n t_i)\ e^{-\sum_{i=1}^n A_i t_i} \\
&= \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_n\ \delta(1 - \sum_{i=1}^n x_i) \\
&\qquad \cdot \int_0^{\infty} ds s^{n-1}\ e^{-s\sum_{i=1}^n A_i x_i} \\
&= (n-1)! \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_n\ \frac{\delta(1 - \sum_{i=1}^n x_i)}{(\sum_{i=1}^n A_i x_i)^n}
\end{aligned}$$

This is just the expected Feynman integral formula.

7. By symmetry consideration, we have:

$$\int d^d x\ x^\mu x^\nu x^\rho x^\sigma f(x^2) = a\eta^{\mu\nu}\eta^{\rho\sigma} + b\eta^{\mu\rho}\eta^{\nu\sigma} + c\eta^{\mu\sigma}\eta^{\rho\nu}$$

Notice that $\eta_{\mu\nu}x^\mu x^\nu = x^2$, $\eta_{\mu\nu}\eta^{\nu\rho} = \delta_\mu^\rho$ and $\eta_{\mu\nu}\eta^{\nu\mu} = d$, we see that $a = b = c$ and

$$\int d^d x\ (x^2)^2 f(x^2) = a(d^2 + 2d) = d(d+2)a$$

This gives:

$$a = \frac{1}{d(d+2)} \int d^d x\ (x^2)^2 f(x^2)$$

Therefore,

$$\int d^d x\; x^\mu x^\nu x^\rho x^\sigma f(x^2) = \left[\frac{\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\rho\nu}}{d(d+2)}\right] \int d^d x\; (x^2)^2 f(x^2)$$

Remember that $(x^2)^2 \neq x^4$. In fact, x^4 has no clear definition.

8. According to Feynman formula,

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[Ax + B(1-x)]^2}$$

Therefore,

$$\frac{1}{(p+k)^2 k^2} = \int_0^1 \frac{dx}{[(p+k)^2 x + k^2(1-x)]^2}$$

Substitution of this formula into the original integral, we get:

$$\begin{aligned} I &= \int \frac{d^d k}{(2\pi)^d} \frac{(p+k)_\mu k_\nu}{(p+k)^2 k^2} \\ &= \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{(p+k)_\mu k_\nu}{[(p+k)^2 x + k^2(1-x)]^2} \end{aligned}$$

To remove the cross term $p \cdot k$ in the denominator of the integrand, we change the variable from k^μ to $q^\mu = k^\mu + xp^\mu$. Hence,

$$\begin{aligned} I &= \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{[q_\mu + (1-x)p_\mu](q_\nu - xp_\nu)}{[q^2 + p^2 x(1-x)]^2} \\ &= \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{q_\mu q_\nu - xq_\mu p_\nu + (1-x)p_\mu q_\nu - x(1-x)p_\mu p_\nu}{[q^2 + p^2 x(1-x)]^2} \\ &= \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{q_\mu q_\nu - x(1-x)p_\mu p_\nu}{[q^2 + p^2 x(1-x)]^2} \\ &= \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{N_{\mu\nu}}{[q^2 + p^2 x(1-x)]^2} \\ &= \int_0^1 dx\; \mathscr{W}(p, x) \end{aligned}$$

where,

$$N_{\mu\nu} = \frac{1}{d}\eta_{\mu\nu}q^2 - x(1-x)p_\mu p_\nu, \qquad \mathscr{W}(p,x) = \int \frac{d^dq}{(2\pi)^d}\frac{N_{\mu\nu}}{[q^2+p^2x(1-x)]^2}$$

In the above calculations we have used the identities

$$\int d^dq\ q_\mu F(q^2) = 0, \qquad \int d^dq\ q_\mu q_\nu F(q^2) = \frac{1}{d}\eta_{\mu\nu}\int d^dq q^2 F(q^2)$$

The calculation of integral $\mathscr{W}(p,x)$ should be performed in d-dimensional Euclidean space[1]. To this end, we make the Wick rotation by setting $q^0 = i\bar{q}_d$, $q^i = \bar{q}_i$ for $(i = 1,\ 2,\ \cdots,\ d-1)$. Thereby,

$$\begin{aligned}\mathscr{W}(p,x) &= i\int \frac{d^d\bar{q}}{(2\pi)^d}\frac{1}{[\bar{q}^2+p^2x(1-x)]^2}\left[\frac{1}{d}\eta_{\mu\nu}\bar{q}^2 - x(1-x)p_\mu p_\nu\right] \\ &= i\frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}\left[\frac{1}{2}\eta_{\mu\nu}D - \left(1-\frac{d}{2}\right)x(1-x)p_\mu p_\nu\right]D^{-(2-d/2)}\end{aligned}$$

where,

$$D = p^2x(1-x)$$

and we have used the formula

$$\int \frac{d^dx}{(2\pi)^d}\frac{(x^2)^a}{(x^2+D)^b} = \frac{\Gamma\left(b-a-\frac{d}{2}\right)\Gamma\left(a+\frac{d}{2}\right)}{(4\pi)^{d/2}\Gamma(b)\Gamma\left(\frac{d}{2}\right)}D^{-(b-a-d/2)}$$

which holds in d-dimensional Euclidean space.

9. Based on the given formula,

$$\begin{aligned}B_n(1-x) &= \left[\frac{\partial^n}{\partial t^n}\left(\frac{te^{(1-x)t}}{e^t-1}\right)\right]_{t=0} \\ &= \left[\frac{\partial^n}{\partial t^n}\left(\frac{te^{-xt}}{1-e^{-t}}\right)\right]_{t=0} \\ &= (-1)^n\left[\frac{\partial^n}{\partial t^n}\left(\frac{te^{xt}}{e^t-1}\right)\right]_{t=0} \\ &= (-1)^n B_n(x)\end{aligned}$$

[1]Otherwise the integral does diverge.

10. According the ζ-regulation scheme,

$$\sum_{n=0}^{+\infty}(n+a)^s = \zeta_R(-s,a) = -\frac{B_{s+1}(a)}{s+1}$$

we see that:

$$\sum_{n=0}^{+\infty}(n+1/3) = -\frac{1}{2}B_2(1/3)$$

Recall that $B_2(x) = x^2 - x + \frac{1}{6}$, we know $B_2(1/3) = -1/18$. Therefore,

$$\sum_{n=0}^{+\infty}(n+1/3) = \frac{1}{36}$$

This result of regulation could also be obtained by using an alternative formula,

$$\sum_{n=1}^{\infty}(n-\theta) = -\frac{1}{12}(6\theta^2 - 6\theta + 1)$$

in which we should take $\theta = 2/3$.

11. Please see the details in my lecture notes.

12. The proof of these formulae requires the use of Poisson resummation formula:

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(y)e^{-2\pi i n y}dy$$

Theta functions with characteristics are defined as,

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\nu,\tau) = \sum_{n=-\infty}^{+\infty} \exp\left[\pi i(n+a)^2\tau + 2\pi i(n+a)(\nu+b)\right]$$

where the parameters a and b take their values of either 0 or 1/2. The alternative notations for them are:

$$\vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\nu,\tau) = \vartheta_{00}(\nu,\tau), \qquad \vartheta\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}(\nu,\tau) = \vartheta_{01}(\nu,\tau),$$

$$\vartheta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(\nu,\tau) = \vartheta_{10}(\nu,\tau), \qquad \vartheta\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}(\nu,\tau) = \vartheta_{11}(\nu,\tau).$$

Let,

$$f(n) = \exp\left[-\frac{\pi i(n+a)^2}{\tau} + \frac{2\pi i(n+a)\nu}{\tau} + 2\pi i(n+a)b\right]$$

We calculate an auxiliary integral:

$$\begin{aligned}
I_n &= \int_{-\infty}^{+\infty} f(y)e^{-2\pi i n y}dy \\
&= \exp(2\pi i n a)\int_{-\infty}^{+\infty} \exp\left[-\frac{\pi i(y+a)^2}{\tau} + 2\pi i(y+a)\Big(\frac{\nu}{\tau}+b-n\Big)\right] dy \\
&= \exp(2\pi i n a)\exp\left[\frac{\pi i}{\tau}(\nu+b\tau-n\tau)^2\right] \\
&\qquad \cdot \int_{-\infty}^{+\infty} dy \exp\left[-\frac{\pi i}{\tau}(y+a-\nu-b\tau+n\tau)^2\right] \\
&= e^{\frac{\pi i\nu^2}{\tau}}\exp(2\pi i n a)\exp[\pi i(n-b)^2\tau - 2\pi i(n-b)\nu] \\
&\qquad \cdot \int_{-\infty}^{+\infty} d\xi \exp\left[-\frac{\pi i}{\tau}\xi^2\right] \\
&= \sqrt{-i\tau}e^{\frac{\pi i\nu^2}{\tau}}\exp(2\pi i a b)\exp[\pi i(n-b)^2\tau - 2\pi i(n-b)(\nu-a)]
\end{aligned}$$

In the last step, we have used the Fresnel integral formula,

$$\int_{-\infty}^{+\infty} e^{itx^2}dx = \int_{-\infty}^{+\infty} e^{-(-it)x^2}dx = \sqrt{\frac{\pi}{-it}} = \sqrt{i\pi/t}$$

Consequently,

$$\begin{aligned}
\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\nu/\tau, -1/\tau) &= \sum_{n=-\infty}^{+\infty} f(n) = \sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(y)e^{-2\pi i n y}dy \\
&= \sum_{n=-\infty}^{+\infty} I_n \\
&= \sqrt{-i\tau}e^{\frac{\pi i\nu^2}{\tau}}\exp(2\pi i a b)\sum_{n=-\infty}^{+\infty}\exp[\pi i(n-b)^2\tau - 2\pi i(n-b)(\nu-a)] \\
&= \sqrt{-i\tau}e^{\frac{\pi i\nu^2}{\tau}}\exp(-2\pi i a b)\sum_{n=-\infty}^{+\infty}\exp[\pi i(n+b)^2\tau + 2\pi i(n+b)(\nu+a)] \\
&= \sqrt{-i\tau}e^{\frac{\pi i\nu^2}{\tau}}\exp(-2\pi i a b)\ \vartheta\begin{bmatrix} b \\ a \end{bmatrix}(\nu, \tau)
\end{aligned}$$

This is the very duality property of Theta functions we have expected.

Chapter 2

Problems:

1. Find the multiplication table for a group with 3 elements and prove that it is unique.
2. Find all essential different possible multiplication tables for groups with 4 elements (which cannot be related by renaming elements).
3. Show that the definition representation of permutation group is reducible.
4. Suppose that D_1 and D_2 are equivalent irreducible representations of a finite group G, such that: $D_2(g) = SD_1(g)S^{-1}, \;\; \forall g \in G$. What can you say about an operator A that satisfies$AD_1(g) = D_2(g)A \; , \;\; \forall g \in G$?
5. Find the group of all the discrete rotations that leave a regular tetrahedron invariant by labeling the four vertices and considering the rotations as permutations on the four vertices. Find the conjugacy classes and the characters of the irreducible representations of this group.

Solutions:

1. We label this group as $G = \{a,\ b,\ e\}$, where e is identity. The product ab must be in G, implying that either $ab = a$ or $ab = e$. If $ab = a$, b would actually be the identity ($b = e$), violating the assumption that G is a group of order 3. Therefore, the unique possibility is:

$$ab = e, \qquad \rightsquigarrow b = a^{-1}$$

The expected multiplication table is:

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

2. The groups are expressed as $G = \{e,\ a,\ b,\ c\}$. An obvious example is $G = \{e,\ a,\ b = a^2,\ c = a^3\}$, which is the Abelian group $\mathscr{Z}_4$. A remarkable characteristic of $\mathscr{Z}_4$ is that all elements have order 4:

$$a^4 = b^4 = c^4 = e$$

The multiplication table of $\mathscr{Z}_4$ is as follows:

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

There is no possibility for G of order 4 to have an Abelian subgroup $\mathscr{Z}_3$. However, the possibility that such a G has several subgroups $\mathscr{Z}_2$ does exist:

$$a^2 = b^2 = c^2 = e$$

In this case, G is also an Abelian group:

$$ab = ba = c, \quad ac = ca = b, \quad bc = cb = a.$$

The corresponding Multiplication table is:

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

3. It is enough to show that the definition representation of S_3 is reducible. In the definition representation of S_n, the "objects" being permuted are the basis vectors of the n-dimensional representation space. Therefore, the representation matrix of group element (jk) is:

$$\langle i|D[(jk)]|l\rangle = \begin{cases} \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}, & \text{if } j = l \text{ or } k = l; \\ \delta_{il} & \text{if } j \neq l \neq k \end{cases}$$

For S_3, these representation matrices can explicitly be written out:

$$D[e] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad D[(12)] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D[(23)] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad D[(31)] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$D[(123)] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad D[(321)] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Besides, S_3 has the following Character table:

	e	(12), (13), (23)	(123), (321)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

The projection operators to the subspaces of these 3 irreducible representations are then:

$$\mathcal{P}_1 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \mathcal{P}_2 = 0, \qquad \mathcal{P}_3 = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

The projection operator $\mathcal{P}_1$ does obviously correspond to an invariant subspace: $D[g]\mathcal{P}_1 = \mathcal{P}_1, \quad \forall g$. The corresponding invariant subspace is:

$$\mathcal{P}_1 |1\rangle = \frac{1}{3}\big[\, |1\rangle + |2\rangle + |3\rangle \,\big] = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Of the definition representation of general S_n, a 1-dimensional invariant subspace is obviously,

$$\frac{1}{n}\big[\, |1\rangle + |2\rangle + \cdots + |n\rangle \,\big]$$

Therefore, the definition representation of S_n is reducible.

4. The equation satisfied by operator A can be recast for an irreducible representation, e.g., for D_1:

$$AD_1(g) = D_2(g)A = [SD_1(g)S^{-1}]A,$$

Namely,

$$(S^{-1}A)D_1(g) = D_1(g)(S^{-1}A)$$

Shur's lemma implies that $S^{-1}A \propto I$. Therefore, $A \propto S$.

5. One characteristic of rotation is that there are some fixed points during the rotations. The rotations that leave a regular tetrahedron invariant must include the following permutations in S_4,

- e, unit element.
- (234) and (243), rotations leaving vertex 1 fixed.
- (134) and (143), rotations leaving vertex 2 fixed.
- (124) and (142), rotations leaving vertex 3 fixed.
- (123) and (132), rotations leaving vertex 4 fixed.

These rotations are assumed to form a group. Consequently, they must include the products of above rotations. Notice that

$$(234)(134) = (14)(23)$$
$$(234)(142) = (12)(34)$$
$$(134)(234) = (13)(24)$$

and other consecutive rotations do not yield new rotations. We conclude that the group of rotations that leave the regular tetrahedron invariant is as below:

$$\begin{aligned} T_4 &= \{e, (123), (132), (134), (143), (124), (142), (234), (243), \\ &\quad (12)(34), (13)(24), (14)(23)\} \end{aligned}$$

whose order is 12. In fact, T_4 is a subgroup of symmetric group S_4, consisting of all even permutations of S_4.

We now try to find congugacy classes. The unit element e alone forms a class, $\mathcal{C}_1 = e$. As for the class of group element (12)(34), we see:

$$\begin{aligned} &[(13)(24)](12)(34)[(24)(13)] = (12)(34) \\ &[(14)(23)](12)(34)[(23)(14)] = (12)(34) \\ &[(123)](12)(34)[(321)] = (14)(23) \\ &[(124)](12)(34)[(421)] = (13)(24) \end{aligned}$$

i.e., $\mathcal{C}_2 = \{(12)(34), (13)(24), (14)(23)\}$ form the second class. For group element (123), we have:

$$\begin{aligned} &[(12)(34)](123)[(34)(12)] = (142) \\ &[(13)(24)](123)[(24)(13)] = (134) \\ &[(14)(23)](123)[(23)(14)] = (243) \\ &[(124)](123)[(421)] = (243) \\ &[(134)](123)[(431)] = (243) \\ &[(234)](123)[(432)] = (134) \\ &[(321)](123)[(123)] = (123) \\ &[(214)](123)[(412)] = (134) \\ &[(314)](123)[(413)] = (142) \\ &[(324)](123)[(423)] = (142) \end{aligned}$$

These equalities imply that $\mathcal{C}_3 = \{(123), (134), (214), (324)\}$ form the third class. Finally, we have:

$$\begin{aligned} &[(12)(34)](124)[(34)(12)] = (132) \\ &[(13)(24)](124)[(24)(13)] = (234) \\ &[(14)(23)](124)[(23)(14)] = (143) \end{aligned}$$

i.e., the group elements $\mathcal{C}_4 = \{(124), (234), (321), (314)\}$ form the fourth class. Therefore, there are ***4 conjugacy classes in total*** in the considered alternating group.

We next calculate the characters of irreducible representations of this group. Having 4 distinct conjugacy classes implies that it has 4 inequivalent irreducible representations. Because $\sum_{i=1}^{4} n_i^2 = 12$, we see that there are 3 *1-dimensional irreducible representations* and a *3-dimensional irreducible representation.* Notice that the group elements in $\mathcal{C}_2$ have order 2 while the elements in $\mathcal{C}_3$ and $\mathcal{C}_4$ have order 3. Besides, the inverses of group elements in $\mathcal{C}_3$ are just in the class $\mathcal{C}_4$. Accurately speaking, there are following nontrivial class multiplication products:

$$\mathcal{C}_2\mathcal{C}_3 = \mathcal{C}_3, \quad \mathcal{C}_2\mathcal{C}_4 = \mathcal{C}_4, \quad \mathcal{C}_3\mathcal{C}_3 = \mathcal{C}_4, \quad \mathcal{C}_4\mathcal{C}_4 = \mathcal{C}_3, \quad \mathcal{C}_3\mathcal{C}_4 = \mathcal{C}_1 \cup \mathcal{C}_2.$$

Taking into account of these factors, we can write down the following unfinished character table:

	χ_1	χ_2	χ_3	χ_4
D_1	1	1	1	1
D_2	1	1	ω	ω^2
D_3	1	1	ω^2	ω
D_4	3	a	b	c

where $\omega = e^{i2\pi/3}$. The orthogonality relations satisfied by these characters further indicate:

$$a = -1, \quad b = c = 0.$$

Chapter 3

Problems:

1. Find all components of matrix $e^{i\alpha A}$ where

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

2. If $[A, B] = B$, calculate $e^{i\alpha A} B e^{-i\alpha A}$.

3. Carry out the expansion of δ_c in Eq.(II.4) to third order in α and β.

Solutions:

1. The generator A satisfies:

$$A^{2n} = E, \qquad A^{2n+1} = A, \qquad (n = 1,\ 2,\ 3,\ \cdots)$$

where,

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore,

$$\begin{aligned} e^{i\alpha A} &= \sum_{n=0}^{+\infty} \frac{(i\alpha)^n}{n!} A^n \\ &= 1 + i\alpha A - \frac{\alpha^2}{2!}E - i\frac{\alpha^3}{3!}A + \frac{\alpha^4}{4!}E + \cdots \\ &= 1 - E + \left[1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} + \cdots\right] E + i\left[\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} + \cdots\right] A \\ &= 1 + (cos\alpha - 1)E + i\sin\alpha A \end{aligned}$$

Alternatively,

$$e^{i\alpha A} = \begin{pmatrix} cos\alpha & 0 & i\sin\alpha \\ 0 & 1 & 0 \\ i\sin\alpha & 0 & cos\alpha \end{pmatrix}$$

2. By mathematical formula,

$$e^A B e^{-A} = B + [A,\ B] + \frac{1}{2!}[A,\ [A,\ B]] + \frac{1}{3!}[A,\ [A,\ [A,\ B]]] + \cdots$$

we have:

$$e^{i\alpha A} B e^{-i\alpha A} = e^{i\alpha} B$$

3. Eq.(II.3) is the multiplication law of Lie group elements,

$$\exp(i\alpha_a X_a)\exp(i\beta_b X_b) = \exp(i\delta_c X_c)$$

To find δ_c, we define a group element which depends on a parameter λ:

$$F(\lambda) = e^{\lambda A} e^{\lambda B} = e^{\lambda C}$$

where $A = i\alpha_a X_a,\ B = i\beta_b X_b$ and $C = i\delta_c X_c$. Obviously, $F(0) = 1$ and $F(1)$ is what we want to evaluate.

The derivative of$F(\lambda)$ with respect to parameterλ is:

$$\begin{aligned} F'(\lambda) &= Ae^{\lambda A}e^{\lambda B} + e^{\lambda A}Be^{\lambda B} \\ &= \left(A + e^{\lambda A}Be^{-\lambda A}\right) F(\lambda) \\ &= \left(A + B + \lambda[A,\ B] + \frac{\lambda^2}{2!}[A,\ [A,\ B]] + \cdots\right) F(\lambda) \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_0^1 d\lambda\ F'(\lambda)F^{-1}(\lambda) \\ &\qquad = \int_0^1 d\lambda\ \left(A + B + \lambda[A,\ B] + \frac{\lambda^2}{2!}[A,\ [A,\ B]] + \cdots\right) \\ &\qquad = A + B + \frac{1}{2}[A,\ B] + \frac{1}{6}[A,\ [A,\ B]] + \cdots \end{aligned}$$

Because $F(\lambda) = e^{\lambda C}$, we have $F'(\lambda)F^{-1}(\lambda) = C$. Consequently,

$$C = A + B + \frac{1}{2}[A,\ B] + \frac{1}{6}[A,\ [A,\ B]] + \cdots$$

Namely,

$$\begin{aligned} i\delta_a X_a &= i\alpha_a X_a + i\beta_a X_a - \frac{1}{2}\alpha_b\beta_c[X_b,\ X_c] - \frac{i}{6}\alpha_b\alpha_c\beta_d[X_b,\ [X_c,\ X_d]] + \cdots \\ &= i\alpha_a X_a + i\beta_a X_a - \frac{i}{2}\alpha_b\beta_c f_{bca}X_a + \frac{i}{6}\alpha_b\alpha_c\beta_d f_{cde}f_{bea}X_a + \cdots \end{aligned}$$

Therefore,

$$\delta_a = \alpha_a + \beta_a - \frac{1}{2}\alpha_b\beta_c f_{bca} + \frac{1}{6}\alpha_b\alpha_c\beta_d f_{cde}f_{bea} + \cdots$$

Chapter 4

Problems:

1. For SU(2), use the highest weight decomposition to show that

$$\{j\} \otimes \{s\} = \bigoplus_{l=|j-s|}^{j+s} \{l\}.$$

2. Calculate $e^{i\vec{r}\cdot\vec{\sigma}}$, where $\vec{\sigma}$ are Pauli matrices.

3. Suppose that $[\sigma_a]_{ij}$ and $[\eta_a]_{xy}$ are Pauli matrices in two different two dimensional spaces. In the 4-d tensor product space, define the basis:

$$|1\rangle = |i=1\rangle\,|x=1\rangle\,, \quad |2\rangle = |i=1\rangle\,|x=2\rangle\,, \quad |3\rangle = |i=2\rangle\,|x=1\rangle\,,$$
$$|4\rangle = |i=2\rangle\,|x=2\rangle\,.$$

   Write out the matrix elements of $\sigma_2 \otimes \eta_1$ in this basis.

4. Ignored.

Solutions:

1. From the highest weight decomposition we know that:

$$l = j+s,\;\; j+s-1,\;\; j+s-2,\;\; j+s-3,\;\cdots\,,\; j+s-n+1.$$

   Thus, to show the above CG decomposition means to calculate the minimum value $l_{\min} = j+s-n+1$ of quantum number l.

For the spin k representation {k} of $SU(2)$, $d_{\{k\}}$=2k+1. The dimension of direct product representation {j}⊗{s} is then: $(2j+1)(2s+1)$. The dimension matching requires that:

$$(2j+1)(2s+1) = \sum_{l=l_{\min}}^{j+s}(2l+1) = n[(j+s)+(j+s-n+1)]+n \quad ,$$

Namely,

$$n^2 - 2(j+s+1)n + (2j+1)(2s+1) = 0,$$

Its solution reads either $n = 2j+1$ or $n = 2s+1$. Consequently, $l_{\min} = |j-s|$.

2. Noticing that $\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$, we get: $(\vec{r}\cdot\vec{\sigma})^2 = r^2$. Therefore,

$$\begin{aligned} e^{i\vec{r}\cdot\vec{\sigma}} &= 1 + i\vec{r}\cdot\vec{\sigma} - \frac{1}{2!}r^2 - \frac{1}{3!}r^2 i\vec{r}\cdot\vec{\sigma} + \cdots \\ &= \cos r + \frac{i\vec{r}\cdot\vec{\sigma}}{r}\sin r \end{aligned}$$

3. Because,

$$\begin{aligned}
\sigma_2\otimes\eta_1|1\rangle &= \sigma_2|i=1\rangle\,\eta_1|x=1\rangle = i\,|i=2\rangle\,|x=2\rangle = i\,|4\rangle \\
\sigma_2\otimes\eta_1|2\rangle &= \sigma_2|i=1\rangle\,\eta_1|x=2\rangle = i\,|i=2\rangle\,|x=1\rangle = i\,|3\rangle \\
\sigma_2\otimes\eta_1|3\rangle &= \sigma_2|i=2\rangle\,\eta_1|x=1\rangle = -i\,|i=1\rangle\,|x=2\rangle = -i\,|2\rangle \\
\sigma_2\otimes\eta_1|4\rangle &= \sigma_2|i=2\rangle\,\eta_1|x=2\rangle = -i\,|i=1\rangle\,|x=1\rangle = -i\,|1\rangle
\end{aligned}$$

we have:

$$\sigma_2\otimes\eta_1 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

This is just $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

4. Because,

$$\begin{aligned}
[\sigma_a,\ \sigma_b\eta_c]_{ix,jy} &= (\sigma_a\sigma_b\eta_c)_{ix,jy} - (\sigma_b\eta_c\sigma_a)_{ix,jy} \\
&= (\sigma_a\sigma_b)_{ij}(\eta_c)_{xy} - (\sigma_b\sigma_a)_{ij}(\eta_c)_{xy} \\
&= [\sigma_a,\ \sigma_b]_{ij}(\eta_c)_{xy} \\
&= 2i\varepsilon_{abd}(\sigma_d)_{ij}(\eta_c)_{xy} \\
&= 2i\varepsilon_{abd}(\sigma_d\eta_c)_{ix,jy}
\end{aligned}$$

we get:

$$[\sigma_a, \sigma_b \eta_c] = 2i\varepsilon_{abd}\sigma_d \eta_c$$

The expected trace is:

$$\begin{aligned} tr(\sigma_a\{\eta_b,\ \sigma_c\eta_d\}) &= (\sigma_a\{\eta_b,\ \sigma_c\eta_d\})_{ix,ix} \\ &= (\sigma_a\eta_b\sigma_c\eta_d + \sigma_a\sigma_c\eta_d\eta_b)_{ix,ix} \\ &= (\sigma_a\sigma_c)_{ii}(\eta_b\eta_d)_{xx} + (\sigma_a\sigma_c)_{ii}(\eta_d\eta_b)_{xx} \\ &= 2tr(\sigma_a\sigma_c)tr(\eta_b\eta_d) \\ &= 8\delta_{ac}\delta_{bd} \end{aligned}$$

The specified commutation relation is vanishing:

$$\begin{aligned} [\sigma_1\eta_1,\ \sigma_2\eta_2]_{ix,jy} &= (\sigma_1\eta_1\sigma_2\eta_2)_{ix,jy} - (\sigma_2\eta_2\sigma_1\eta_1)_{ix,jy} \\ &= (\sigma_1\sigma_2)_{ij}(\eta_1\eta_2)_{xy} - (\sigma_2\sigma_1)_{ij}(\eta_2\eta_1)_{xy} \\ &= (\sigma_1\sigma_2)_{ij}(\eta_1\eta_2)_{xy} - (\sigma_1\sigma_2)_{ij}(\eta_1\eta_2)_{xy} \\ &= 0. \end{aligned}$$

Chapter 5

Problems:

1. Consider an operator O_x, for $x = 1$ to 2, transforming according to the spin 1/2 representation of $SU(2)$ as follows:
$$[J_a,\ \ O_x] = O_y[\sigma_a]_{yx}/2.$$
Given$\langle 3/2,\ -1/2,\ \alpha|\, O_1\, |1,\ -1,\ \beta\rangle$ = A, find
$$\langle 3/2,\ -3/2,\ \alpha|\, O_2\, |1,\ -1,\ \beta\rangle = ?$$

2. The operator $(\mathrm{x}_{+1})^2$ satisfies
$$\left[L_+,\ (x_{+1})^2\right] = 0.$$
It is therefore the O_{+2} components of a spin-2 tensor operator. Construct the other components, O_m. Note that the product of tensor operators transforms like the tensor product of their representations. What is the connection of these with the spherical harmonics, $\mathscr{Y}_{lm}(\theta,\ \phi)$?

Solutions:

1. According to Wigner-Eckart theorem, $O_1 \equiv O^{1/2}_{1/2}$, $O_2 \equiv O^{1/2}_{-1/2}$, we recast the given condition as:
$$\begin{aligned} A &= \langle 3/2,\ -1/2,\ \alpha|\, O_1\, |1,\ -1,\ \beta\rangle \\ &= \langle 3/2,\ -1/2 \mid 1/2,\ 1,\ 1/2,\ -1\rangle \cdot \langle 3/2,\ \alpha|\, O^{1/2}\, |1,\ \beta\rangle \\ &= \sqrt{\tfrac{1}{3}} \cdot \langle 3/2,\ \alpha|\, O^{1/2}\, |1,\ \beta\rangle \end{aligned}$$

We have used the CG coefficient determined by,

$$|3/2,\ -1/2\rangle = \sqrt{\frac{2}{3}}\,|1/2,\ 1,\ -1/2,\ 0\rangle + \sqrt{\frac{1}{3}}\,|1/2,\ 1,\ 1/2,\ -1\rangle\,.$$

Similarly,

$$|3/2,\ -3/2\rangle = |1/2,\ 1,\ -1/2,\ -1\rangle\,.$$

Therefore:

$$\begin{aligned}
&\langle 3/2,\ -3/2,\ \alpha|\,O_2\,|1,\ -1,\ \beta\rangle = \langle 3/2,\ -3/2,\ \alpha|\,O^{1/2}_{-1/2}\,|1,\ -1,\ \beta\rangle \\
&= \langle 3/2,\ -3/2\mid 1/2,\ 1,\ -1/2,\ -1\rangle\cdot\langle 3/2,\ \alpha|\,O^{1/2}\,|1,\ \beta\rangle \\
&= \langle 3/2,\ \alpha|\,O^{1/2}\,|1,\ \beta\rangle \\
&= \sqrt{3}\,A
\end{aligned}$$

2. In the spherical system of coordinate representation,

$$x_1 = r\sin\theta\cos\varphi,\quad x_2 = r\sin\theta\sin\varphi,\quad x_3 = r\cos\theta.$$

i.e.,

$$\begin{aligned}
x_{\pm1} &= \mp\frac{1}{\sqrt{2}}(x_1 \pm i x_2) = \mp\frac{r}{\sqrt{2}}\sin\theta e^{\pm i\varphi} \\
x_3 &= r\cos\theta
\end{aligned}$$

The orbital angular momentum is defined as:

$$\begin{aligned}
\vec{L} &= -i\vec{x}\times\nabla \\
&= -ir\vec{e}_r\times\left[\vec{e}_r\partial_r + \frac{1}{r}\vec{e}_\theta\partial_\theta + \frac{1}{r\sin\theta}\vec{e}_\varphi\partial_\varphi\right] \\
&= -i\left[\vec{e}_\varphi\partial_\theta - \frac{1}{\sin\theta}\vec{e}_\theta\partial_\varphi\right]
\end{aligned}$$

Because,

$$\vec{e}_r = \frac{\vec{r}}{r} = \vec{i}\sin\theta\cos\varphi + \vec{j}\sin\theta\sin\varphi + \vec{k}\cos\theta,$$

we see that

$$\begin{aligned}
\vec{e}_\theta &= \frac{\partial\vec{e}_r}{\partial\theta} = \vec{i}\cos\theta\cos\varphi + \vec{j}\cos\theta\sin\varphi - \vec{k}\sin\theta \\
\vec{e}_\varphi &= \frac{1}{\sin\theta}\frac{\partial\vec{e}_r}{\partial\varphi} = -\vec{i}\sin\varphi + \vec{j}\cos\varphi
\end{aligned}$$

Thus,

$$\vec{L} = -i\left[\left(-\vec{i}\sin\varphi + \vec{j}\cos\varphi\right)\partial_\theta - \frac{1}{\sin\theta}\left(\vec{i}\cos\theta\cos\varphi + \vec{j}\cos\theta\sin\varphi - \vec{k}\sin\theta\right)\partial_\varphi\right]$$

i.e.,

$$\begin{aligned} L_1 &= -i\left[-\sin\varphi\partial_\theta - ctg\theta\cos\varphi\partial_\varphi\right] \\ L_2 &= -i\left[\cos\varphi\partial_\theta - ctg\theta\sin\varphi\partial_\varphi\right] \\ L_3 &= -i\partial_\varphi \end{aligned}$$

and

$$\begin{aligned} L_\pm &= \frac{1}{\sqrt{2}}\left(L_1 \pm iL_2\right) \\ &= -\frac{i}{\sqrt{2}}\left[(-\sin\varphi \pm i\cos\varphi)\partial_\theta - ctg\theta(\cos\varphi \pm i\sin\varphi)\partial_\varphi\right] \\ &= \frac{e^{\pm i\varphi}}{\sqrt{2}}(\pm\partial_\theta + ictg\theta\partial_\varphi) \end{aligned}$$

Let us make a check that these angular momentum operators obey the $SU(2)$ algebra:

$$\begin{aligned} [L_+,\ L_-] &= \frac{1}{2}\left[e^{i\varphi}(\partial_\theta + ictg\theta\partial_\varphi),\ \ e^{-i\varphi}(-\partial_\theta + ictg\theta\partial_\varphi)\right] \\ &= \frac{1}{2}\left[e^{i\varphi}\partial_\theta,\ e^{-i\varphi}ictg\theta\partial_\varphi\right] + \frac{1}{2}\left[e^{i\varphi}ictg\theta\partial_\varphi,\ -e^{-i\varphi}\partial_\theta\right] \\ &\qquad -\frac{1}{2}ctg^2\theta\left[e^{i\varphi}\partial_\varphi,\ e^{-i\varphi}\partial_\varphi\right] \\ &= -i\partial_\varphi \\ &= L_3 \\ [L_3,\ L_\pm] &= \left[-i\partial_\varphi,\ \ \frac{e^{\pm i\varphi}}{\sqrt{2}}(\pm\partial_\theta + ictg\theta\partial_\varphi)\right] \\ &= \pm\frac{e^{\pm i\varphi}}{\sqrt{2}}(\pm\partial_\theta + ictg\theta\partial_\varphi) \\ &= \pm L_\pm\ . \end{aligned}$$

Noticing $(x_{\pm 1})^2 = \frac{1}{2} r^2 \sin^2\theta e^{\pm 2i\varphi}$,   we get:

$$
\begin{aligned}
[L_+,\ (x_{+1})^2] &= \frac{r^2}{2\sqrt{2}} \left[e^{i\varphi}(\partial_\theta + i ctg\theta \partial_\varphi),\ \ e^{2i\varphi}\sin^2\theta \right] \\
&= \frac{r^2}{2\sqrt{2}} e^{3i\varphi} \left[2\sin\theta\cos\theta + 2i^2 ctg\theta \sin^2\theta \right] \\
&= 0.
\end{aligned}
$$

This is just the assumption in the statements of the problem. Other components of the tensor are calculated as:

$$
\begin{aligned}
[L_-,\ (x_{+1})^2] &= \frac{r^2}{2\sqrt{2}} \left[e^{-i\varphi}(-\partial_\theta + i ctg\theta \partial_\varphi),\ \ e^{2i\varphi}\sin^2\theta \right] \\
&= \frac{r^2}{2\sqrt{2}} e^{i\varphi} \left[-2\sin\theta\cos\theta + 2i^2 ctg\theta \sin^2\theta \right] \\
&= -\sqrt{2} r^2 \sin\theta\cos\theta e^{i\varphi} \\
&= 2x_3 x_{+1}
\end{aligned}
$$

$$
\begin{aligned}
[L_-,\ 2x_3 x_{+1}] &= -\frac{r^2}{\sqrt{2}} \left[e^{-i\varphi}(-\partial_\theta + i ctg\theta \partial_\varphi),\ \ \sqrt{2} e^{i\varphi}\sin\theta\cos\theta \right] \\
&= -r^2 \left[-\cos^2\theta + \sin^2\theta - ctg\theta\sin\theta\cos\theta \right] \\
&= -r^2 (1 - 3\cos^2\theta) \\
&= 2(x_{+1}x_{-1} + x_3^2)
\end{aligned}
$$

$$
\begin{aligned}
[L_-,\ 2(x_{+1}x_{-1} + x_3^2)] &= \frac{6r^2}{\sqrt{2}} \cos\theta\sin\theta e^{-i\varphi} \\
&= 6x_3 x_{-1}
\end{aligned}
$$

$$
\begin{aligned}
[L_-,\ 6x_3 x_{-1}] &= 3r^2 \left[e^{-i\varphi}(-\partial_\theta + i ctg\theta \partial_\varphi),\ \ \cos\theta\sin\theta e^{-i\varphi} \right] \\
&= 3r^2 e^{-2i\varphi} \left[\sin^2\theta - \cos^2\theta + ctg\theta\cos\theta\sin\theta \right] \\
&= 3r^2 \sin^2\theta e^{-2i\varphi} \\
&= 6(x_{-1})^2
\end{aligned}
$$

Apart from some unimportant coefficients, these tensor components are just the spherical harmonics $\mathscr{Y}_{2m}(\theta,\ \phi)$ for $m = 0,\ \pm 1$ and ± 2.

Chapter 6

Problems:

1. Show that $[E_\alpha,\ E_\beta]$ must be proportional to $E_{\alpha+\beta}$. What happens if $\alpha+\beta$ is not a root ?

2. Suppose that the raising lowering operators of some Lie algebra g satisfy

$$[E_\alpha,\ \ E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$$

   for some nonzero coefficients$N_{\alpha,\beta}$. Calculate $[E_\alpha,\ \ E_{-\alpha-\beta}]$.

3. Consider the simple Lie algebra formed by the 10 matrices:

$$\sigma_a/2, \quad \sigma_a\tau_1/2, \quad \sigma_a\tau_3/2, \quad \tau_2/2$$

   for $a = 1$ to 3 where σ_a and τ_a are Pauli matrices in orthogonal spaces.

   - Show that these 10 matrices generate the spinor representation of $SO(5)$.
   - Take $H_1 = \sigma_3/2$ and $H_2 = \sigma_3\tau_3/2$ as the Cartan subalgebra. Find the weights of this spinor representation.
   - Find the roots of the adjoint representation.

Solutions:

1. If both α and β are roots, we have $[H_i,\ E_\alpha] = \alpha_i E_\alpha, \quad [H_i,\ E_\beta] = \beta_i E_\beta$. If their sum is also a root, we have further,

$$[H_i,\ E_{\alpha+\beta}] = (\alpha+\beta)_i E_{\alpha+\beta}.$$

On the other hand, we have the following Jacobean identity:

$$0 = [H_i,\ \ [E_\alpha,\ \ E_\beta]] + [E_\beta,\ \ [H_i,\ \ E_\alpha]] + [E_\alpha,\ \ [E_\beta,\ \ H_i]],$$

i.e.,

$$0 = [H_i,\ \ [E_\alpha,\ \ E_\beta]] + \alpha_i [E_\beta,\ \ E_\alpha] - \beta_i [E_\alpha,\ \ E_\beta].$$

Therefore,

$$[H_i,\ \ [E_\alpha,\ \ E_\beta]] = (\alpha+\beta)_i [E_\alpha,\ \ E_\beta], \qquad \rightsquigarrow\ [E_\alpha,\ \ E_\beta] = N_{\alpha+\beta} E_{\alpha+\beta}.$$

When $\alpha+\beta$ is not a root, the corresponding $E_{\alpha+\beta}$ does not exist. Therefore,

$$[E_\alpha,\ \ E_\beta] = 0.$$

if $\alpha+\beta$ is not a root vector.

2. In terms of the definition we have $N_{\alpha,\beta} = -N_{\beta,\alpha}$. Besides, the generators related to $\pm\alpha$ are hermitian conjugate one another: $(E_\alpha)^+ = E_{-\alpha}$. This indicates that

$$E_{-\alpha-\beta} = -\frac{[E_{-\alpha},\ \ E_{-\beta}]}{(N_{\alpha,\beta})*}$$

Consequently,

$$\begin{aligned}[E_\alpha,\ \ E_{-\alpha-\beta}] &= -[E_\alpha,\ [E_{-\alpha},\ \ E_{-\beta}]]/(N_{\alpha,\beta})* \\ &= ([E_{-\alpha},\ [E_{-\beta},\ \ E_\alpha]] + [E_{-\beta},\ [E_\alpha,\ E_{-\alpha}]])/(N_{\alpha,\beta})* \\ &= -N_{\alpha,-\beta}[E_{-\alpha},\ E_{\alpha-\beta}]/(N_{\alpha,\beta})* + [E_{-\beta},\ \alpha\cdot H]/(N_{\alpha,\beta})* \\ &= [(N_{\alpha,-\beta}N_{\alpha-\beta,-\alpha} + \alpha\cdot\beta)/(N_{\alpha,\beta})*]\, E_{-\beta}\end{aligned}$$

3. The given Lie algebra has 10 generators, which coincides with those of $SO(5)$. Let,

$$\begin{aligned}&M_{1,5} = \sigma_1/2, \quad M_{2,5} = \sigma_2/2, \quad M_{3,5} = \sigma_3\tau_3/2, \\ &M_{4,5} = \sigma_3\tau_1/2,\end{aligned}$$

We get:

$$
\begin{aligned}
M_{1,2} &= -i[M_{1,5},\ \ M_{2,5}] = \sigma_3/2, \\
M_{1,3} &= -i[M_{1,5},\ \ M_{3,5}] = -\sigma_2\tau_3/2, \\
M_{1,4} &= -i[M_{1,5},\ \ M_{4,5}] = -\sigma_2\tau_1/2, \\
M_{2,3} &= -i[M_{2,5},\ \ M_{3,5}] = \sigma_1\tau_1/2, \\
M_{2,4} &= -i[M_{2,5},\ \ M_{4,5}] = \sigma_1\tau_1/2, \\
M_{3,4} &= -i[M_{3,5},\ \ M_{4,5}] = \tau_2/2
\end{aligned}
$$

These $M_{a,b}$ satisfy the algebra of $SO(5)$:

$$[M_{a,b},\ \ M_{c,d}] = -i(\delta_{bc}M_{a,d} - \delta_{ac}M_{b,d} - \delta_{bd}M_{a,c} + \delta_{ad}M_{b,c})$$

Recalling that they are 4-dimensional matrices, they give the spinor representation of $SO(5)$.

If we denote

$$H_1 = \sigma_3/2, \quad H_2 = \sigma_3\tau_3/2,$$

we see:

$$H_1 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad H_2 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The weights of this spinor representation are then:

$$
\begin{aligned}
&\nu_1 = (1/2,\ \ 1/2)\,,\ \ \nu_2 = (1/2,\ \ -1/2)\,, \\
&\nu_3 = (-1/2,\ \ -1/2)\,,\ \ \nu_4 = (-1/2,\ \ 1/2)\,.
\end{aligned}
$$

Noticing that the number of positive roots is the same as half of those of the non-Cartan generators, we know that there are 4 positive roots in $SO(5)$. Because the differences between the weights in any representation are probably the roots of the algebra, we guess the following candidates for $SO(5)$ positive roots :

$$\alpha_1 = (1,\ \ -1)\,,\ \ \alpha_2 = (0,\ \ 1)\,,\ \ \alpha_3 = (1,\ \ 0)\,,\ \ \alpha_4 = (1,\ \ 1)\,.$$

Since,

$$\alpha_3 = \alpha_1 + \alpha_2, \quad \alpha_4 = \alpha_1 + 2\alpha_2$$

we see that $\alpha_1,\ \alpha_2$ are simple roots.

Chapter 7

Problems:

1. Calculate f_{147} and f_{458} in $SU(3)$.

2. Show that T_1, T_2 and T_3 generate an $SU(2)$ sub-algebra of $SU(3)$. How does the representation generated by (VII.1 and 2) transform under this sub-algebra ?

Solutions:

1. The generators T_1, T_4 and T_5 are:

$$T_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad T_4 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad T_5 = \frac{1}{2}\begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}$$

So,

$$
\begin{aligned}
[T_1,\ T_4] &= \frac{1}{4}\left\{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\} \\
&= \frac{1}{4}\left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right\} \\
&= \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \\
&= \frac{i}{2}\cdot\frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \\
&= \frac{i}{2}T_7
\end{aligned}
$$

we see $f_{147} = 1/2$. Similarly,

$$
\begin{aligned}
[T_4,\ T_5] &= \frac{1}{4}\left\{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right\} \\
&= \frac{1}{4}\left\{\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{bmatrix}\right\} \\
&= \frac{1}{2}\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{bmatrix}
\end{aligned}
$$

It is known that:

$$
T_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad T_8 = \frac{1}{2\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}
$$

Let

$$
\frac{1}{2}\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{bmatrix} = aT_3 + bT_8
$$

Because $tr(T_aT_b) = \frac{1}{2}\delta_{ab}$, we see:

$$\begin{aligned} a &= 2tr\left\{\frac{1}{2}\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{bmatrix} T_3\right\} = \frac{1}{2}tr\left\{\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\} \\ &= \frac{1}{2}tr\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \frac{i}{2} \end{aligned}$$

$$\begin{aligned} b &= 2tr\left\{\frac{1}{2}\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{bmatrix} T_8\right\} = \frac{1}{2\sqrt{3}}tr\left\{\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\right\} \\ &= \frac{1}{2\sqrt{3}}tr\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2i \end{bmatrix} \\ &= \frac{\sqrt{3}}{2}i \end{aligned}$$

It indicates that $f_{458} = \frac{\sqrt{3}}{2}$.

2. Define

$$T_3 = \frac{\lambda_3}{2}, \qquad T_\pm = \frac{1}{2\sqrt{2}}\left[\lambda_1 \pm i\lambda_2\right]$$

We see:

$$T_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T_+ = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T_- = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is obviously that:

$$[T_3,\ T_\pm] = \pm T_\pm, \qquad [T_+,\ T_-] = T_3.$$

This is the $SU(2)$ algebra in the standard basis. Because

$$T_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \oplus 0,$$

$$T_+ = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus 0,$$

$$T_- = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \oplus 0.$$

we see that the representation defined as Eqs.(VII.1 and 2) transforms as the direct sum of irreducible representations $j = 1/2$ and $j = 0$.

Chapter 8

Problems:

1. Find the simple roots and fundamental weights and the Dynkin diagram for the algebra discussed in problem (VI.3).

2. Consider the algebra generated by σ_a and $\sigma_a \eta_1$ where σ_a and η_a are independent Paili matrices. Show that this algebra generates a group which is semisimple but not simple. Nevertheless, you can define simple roots. What does the Dynkin diagram look like ?

Solutions:

1. Among the 4 positive roots of $SO(5)$ founded in problem (VI.3),

$$\alpha_1 = (1,\ \ -1)\,,\ \ \alpha_2 = (0,\ \ 1)\,,\ \ \alpha_3 = (1,\ \ 0)\,,\ \ \alpha_4 = (1,\ \ 1)\,.$$

we have seen that:

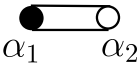
$$\alpha_3 = \alpha_1 + \alpha_2, \quad \alpha_4 = \alpha_1 + 2\alpha_2$$

so α_1 and α_2 are simple roots. Obviously,

$$(\alpha_1)^2 = 2,\ \ (\alpha_2)^2 = 1,\ \ \alpha_1 \cdot \alpha_2 = -1.$$

$$\cos\theta_{12} = \frac{\alpha_1 \cdot \alpha_2}{\sqrt{(\alpha_1)^2}\sqrt{(\alpha_2)^2}} = -\frac{1}{\sqrt{2}}.$$

This implies that θ_{12}=135^0. Consequently, the Dynkin diagram of the algebra $SO(5)$ is:



By using the definition of the fundamental weights,

$$\frac{2\alpha_i \cdot M_j}{\alpha_i^2} = \delta_{ij}, \quad (i,\ j = 1,\ 2)$$

we get the fundamental weights of this algebra as follows:

$$M_1 = (1,\ \ 0)\,, \quad M_2 = (1/2,\ \ 1/2)\,.$$

2. This algebra is defined by the following commutation relations:

$$\begin{aligned}
&[\sigma_a,\ \sigma_b] = 2i\varepsilon_{abc}\sigma_c, \quad [\sigma_a,\ \sigma_b\eta_1] = 2i\varepsilon_{abc}\sigma_c\eta_1,\\
&[\sigma_a\eta_1,\ \sigma_b\eta_1] = (\sigma_a \otimes \eta_1)(\sigma_b \otimes \eta_1) - (\sigma_b \otimes \eta_1)(\sigma_a \otimes \eta_1)\\
&\qquad = \sigma_a\sigma_b \otimes \eta_1^2 - \sigma_b\sigma_a \otimes \eta_1^2\\
&\qquad = [\sigma_a,\ \sigma_b] \otimes 1 = 2i\varepsilon_{abc}\sigma_c.
\end{aligned}$$

So, there is a non-abelian invariant sub-algebras in it, generated by generators $\sigma_a + \sigma_a\eta_1$:

$$\begin{aligned}
&[\sigma_a + \sigma_a\eta_1,\ \sigma_b + \sigma_b\eta_1] = 4i\varepsilon_{abc}(\sigma_c + \sigma_c\eta_1),\\
&[\sigma_a + \sigma_a\eta_1,\ \sigma_b] = 2i\varepsilon_{abc}(\sigma_c + \sigma_c\eta_1),\\
&[\sigma_a + \sigma_a\eta_1,\ \sigma_b\eta_1] = 2i\varepsilon_{abc}(\sigma_c + \sigma_c\eta_1).
\end{aligned}$$

Instead of being simple, It is semi-simple. In physics, it is nothing but the algebra of Lorentz group $SO(1,3)$ [or rotation group $SO(4)$] in 4-dimensions.

In a representation of this algebra $SO(1,3)$, we can assume that:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix};$$

$$\eta_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \eta_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \eta_3 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

From the commutation relations we see that the rank of $SO(1,3)$ is 2. The matrices of Cartan generators in the considered representation are:

$$H_1 = \frac{1}{2}\sigma_3 \otimes 1 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$H_2 = \frac{1}{2}\sigma_3 \otimes \eta_1 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The weights of this representation read: $\nu_1 = (1/2,\ 1/2)\,,\quad \nu_2 = (1/2,\ -1/2)\,,\quad \nu_3 = (-1/2,\ -1/2)\,,\quad \nu_4 = (-1/2,\ 1/2)\,.$ If this were an irreducible representation of $SO(1,3)$, we would obtain the following positive roots of the algebra from the differences of the above weights:

$$\alpha_1 = (1,\ -1)\,,\ \ \alpha_2 = (0,\ 1)\,,\ \ \alpha_3 = (1,\ 0)\,,\ \ \alpha_4 = (1,\ 1)\,.$$

However, it is not the case because $SO(1,3)$ has 6 generators. In fact, the representation under consideration is not irreducible. It is a direct sum of two irreducible spinor representations of $SO(1,3)$ [Recalling quantum field theory for Dirac particle]. The weights of these two irreducible spinor representations are

$$M_1 = (1/2,\ 1/2)\,,\ \ M_2 = (-1/2,\ -1/2)$$

and

$$M_1' = (1/2,\ -1/2)\,,\ \ M_2' = (-1/2,\ 1/2)$$

respectively. Consequently, there are only 2 positive roots in $SO(1,3)$:

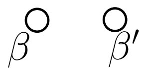
$$\beta = M_1 - M_2 = (1,\ 1),$$

$$\beta' = M_1' - M_2' = (1,\ -1).$$

They are also the simple roots of $SO(1,3)$. Because $\beta'^2 = \beta^2 = 2$ and

$$\cos\theta_{\beta\beta'} = \frac{\beta\cdot\beta'}{\sqrt{\beta^2\beta'^2}} = 0, \quad \theta_{\beta\beta'} = \frac{\pi}{2},$$

the Dynkin diagram of this algebra should be:



An alternative method of determining the roots of the algebra is to directly calculate the commutation relations between the generators in canonical basis, in which $[H_i,\ E_\alpha] = \alpha_i E_\alpha$. Firstly, we have to formulate the generators σ_a and $\sigma_a \eta_1$ in the canonical basis:

$$
\begin{aligned}
H_1 &= \frac{1}{2}\sigma_3 \otimes (1+\eta_1), \quad H_2 = \frac{1}{2}\sigma_3 \otimes (1-\eta_1), \\
E_{\alpha_1} &= \frac{1}{2}(\sigma_1 + i\sigma_2) \otimes (1+\eta_1), \\
E_{\alpha_2} &= \frac{1}{2}(\sigma_1 + i\sigma_2) \otimes (1-\eta_1), \\
E_{-\alpha_1} &= \frac{1}{2}(\sigma_1 - i\sigma_2) \otimes (1+\eta_1), \\
E_{-\alpha_2} &= \frac{1}{2}(\sigma_1 - i\sigma_2) \otimes (1-\eta_1).
\end{aligned}
$$

Then,

$$
\begin{aligned}
[H_1,\ E_{\pm\alpha_1}] &= \frac{1}{4}[\sigma_3,\ \sigma_1 \pm i\sigma_2] \otimes (1+\eta_1)^2 \\
&= \frac{1}{2}(2i\sigma_2 \mp i2i\sigma_1) \otimes (1+\eta_1) \\
&= \pm(\sigma_1 \pm i\sigma_2) \otimes (1+\eta_1) \\
&= \pm 2E_{\pm\alpha_1}
\end{aligned}
$$

$$
\begin{aligned}
[H_2,\ E_{\pm\alpha_1}] &= \frac{1}{4}\sigma_3(\sigma_1 \pm i\sigma_2) \otimes (1-\eta_1)(1+\eta_1) - \frac{1}{4}(\sigma_1 \pm i\sigma_2)\sigma_3 \otimes (1+\eta_1)(1-\eta_1) \\
&= 0\ .
\end{aligned}
$$

Similarly,

$$[H_1,\ E_{\pm\alpha_2}] = 0, \quad [H_2,\ E_{\pm\alpha_2}] = \pm 2E_{\pm\alpha_2}.$$

These commutation relations imply that the positive roots are α_1 and α_2, which are also the simple roots of this semi-simple algebra:

$$
\begin{aligned}
&\alpha_1 = (2,\ \ 0)\,, \\
&\alpha_2 = (0,\ \ 2)\,. \\
&\cos\theta_{\alpha_1\alpha_2} = \frac{\alpha_1 \cdot \alpha_2}{\sqrt{\alpha_1^2 \alpha_2^2}} = 0, \\
&\theta_{\alpha_1\alpha_2} = \frac{\pi}{2}\,.
\end{aligned}
$$

Chapter 9

Problems:

1. Consider the following matrices defined in the 6-dimensional tensor product space of the $SU(3)$ λ_a matrices and the Pauli matricesgσ_a: $\frac{1}{2}\lambda_a\sigma_2$ for a=1, 3, 4, 6 and 8; $\frac{1}{2}\lambda_a$ for a=2, 5 and 7. Show that these generators generate a reducible representation and reduce it.

2. Decompose the tensor product of $\mathbf{3} \times \mathbf{3}$ of $SU(3)$ using highest weight techniques.

Solutions:

1. The Gell-mann matrices λ_a satisfy the $SU(3)$ Lie algebra,

$$[\lambda_a,\ \lambda_b] = 2if_{abc}\lambda_c$$

Consequently,

$$\left[\frac{1}{2}\lambda_a \otimes \sigma_2,\ \frac{1}{2}\lambda_b \otimes \sigma_2\right] = \frac{1}{4}\left[\lambda_a,\ \lambda_b\right] \otimes 1 = if_{abc}\left(\frac{1}{2}\lambda_c \otimes 1\right)$$
$$\left[\frac{1}{2}\lambda_a \otimes \sigma_2,\ \frac{1}{2}\lambda_b \otimes 1\right] = \frac{1}{4}\left[\lambda_a,\ \lambda_b\right] \otimes \sigma_2 = if_{abc}\left(\frac{1}{2}\lambda_c \otimes \sigma_2\right)$$
$$\left[\frac{1}{2}\lambda_a \otimes 1,\ \frac{1}{2}\lambda_b \otimes 1\right] = \frac{1}{4}\left[\lambda_a,\ \lambda_b\right] \otimes 1 = if_{abc}\left(\frac{1}{2}\lambda_c \otimes 1\right)$$

It means that these matrices form a representation of $SU(3)$. The representation matrices for Cartan generators are:

$$\begin{aligned} H_1 &= \frac{1}{2}\lambda_3 \otimes \sigma_2 \\ &= \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} H_2 &= \frac{1}{2}\lambda_8 \otimes \sigma_2 \\ &= \frac{1}{2\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \frac{1}{2\sqrt{3}}\begin{bmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & -2 & \\ & & & & & 2 \end{bmatrix} \end{aligned}$$

The weight vectors of this 6-dimensional representation read:

$$\vec{\mu} = \left\{ \begin{array}{l} \left(1/2,\ 1/2\sqrt{3}\right),\ \left(1/2,\ -1/2\sqrt{3}\right),\ \left(-1/2,\ 1/2\sqrt{3}\right), \\ \left(-1/2,\ -1/2\sqrt{3}\right),\ \left(0,\ -1/\sqrt{3}\right),\ \left(0,\ 1/\sqrt{3}\right) \end{array} \right\}$$

They don't coincide with those of the 6-dimensional irreducible representations $(2,0)$ or $(0,2)$ of $SU(3)$. So it provides for $SU(3)$ a reducible representation.

Notice that the above weight vectors can be divided into two sets:

$$\vec{\mu} = \vec{\mu}_1 + \vec{\mu}_2$$

where,

$$\vec{\mu}_1 = \left\{ \left(1/2,\ 1\Big/2\sqrt{3}\right),\ \left(-1/2,\ 1\Big/2\sqrt{3}\right),\ \left(0,\ -1\Big/\sqrt{3}\right) \right\}$$
$$\vec{\mu}_2 = \left\{ \left(1/2,\ -1\Big/2\sqrt{3}\right),\ \left(0,\ 1\Big/\sqrt{3}\right),\ \left(-1/2,\ -1\Big/2\sqrt{3}\right) \right\}$$

the considered reducible representation of $SU(3)$ can be decomposed into the direct sum of two fundamental representations $(1,0) = \mathbf{3}$ and $(0,1) = \bar{\mathbf{3}}$.

2. The Cartan generators of representation $\mathbf{3} \times \mathbf{3}$ are:

$$\begin{aligned} H_1^{3\times3} &= H_1 \otimes 1 + 1 \otimes H_1 \\ &= \begin{bmatrix} 1/2 & & \\ & -1/2 & \\ & & 0 \end{bmatrix} \otimes 1 + 1 \otimes \begin{bmatrix} 1/2 & & \\ & -1/2 & \\ & & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} H_2^{3\times3} &= H_2 \otimes 1 + 1 \otimes H_2 \\ &= \begin{bmatrix} 1/2\sqrt{3} & & \\ & 1/2\sqrt{3} & \\ & & -1/\sqrt{3} \end{bmatrix} \otimes 1 + 1 \otimes \begin{bmatrix} 1/2\sqrt{3} & & \\ & 1/2\sqrt{3} & \\ & & -1/\sqrt{3} \end{bmatrix} \end{aligned}$$

Therefore, the weight vectors of representation $\mathbf{3} \times \mathbf{3}$ are as follows:

$$\begin{aligned} &\vec{\mu}_1 = \left(1,\ 1\Big/\sqrt{3}\right), \quad \vec{\mu}_2 = \left(0,\ 1\Big/\sqrt{3}\right), \quad \vec{\mu}_3 = \left(1/2,\ -1\Big/2\sqrt{3}\right), \\ &\vec{\mu}_4 = \left(0,\ 1\Big/\sqrt{3}\right), \quad \vec{\mu}_5 = \left(-1,\ 1\Big/\sqrt{3}\right), \\ &\vec{\mu}_6 = \left(-1/2,\ -1\Big/2\sqrt{3}\right), \quad \vec{\mu}_7 = \left(1/2,\ -1\Big/2\sqrt{3}\right), \\ &\vec{\mu}_8 = \left(-1/2,\ -1\Big/2\sqrt{3}\right), \quad \vec{\mu}_9 = \left(0,\ -2\Big/\sqrt{3}\right). \end{aligned}$$

These weights can be divided into two sets:

$$\mathscr{S}_1 = \{\vec{\mu}_2,\ \vec{\mu}_3,\ \vec{\mu}_6\};$$
$$\mathscr{S}_2 = \{\vec{\mu}_1,\ \vec{\mu}_4,\ \vec{\mu}_5,\ \vec{\mu}_7,\ \vec{\mu}_8,\ \vec{\mu}_9\}$$

By examining the weight vectors of irreducible representations of $SU(3)$ [Please see my lecture note for explanation], we see that: $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}$.

Chapter 10

Problems:

1. Decompose the product of tensor components $u^i v^{jk}$, where $v^{jk} = v^{kj}$ transforms like a tensor in Rep.**6** of $SU(3)$.
2. Find the matrix elements $\langle u|X_a|v\rangle$, where X_a stand for the $SU(3)$ generators and $|u\rangle$ and $|v\rangle$ are states in the adjoint representation of $SU(3)$ with tensor components u^i_j and v^i_j. Write the result in terms of the tensor components and the Gell-Mann Matrices.
3. In Rep. **6** of $SU(3)$, for each weight find the corresponding tensor component v^{ij}.
4. Find $(2,1)\otimes(2,1)$ for $SU(3)$. Can you determine which representations appear anti-symmetrically in the tensor product, and which appear symmetrically?
5. Find $\mathbf{10} \times \mathbf{8}$.
6. For any Lie group, the tensor product of the adjoint representation with any arbitrary nontrivial representation D must contain D (think about the action of the generators on the states of D and see if you can figure out why this is so.). In particular, you know that for any nontrivial $SU(3)$ representation D. How can you see this using Young tableaux?

Solutions:

1. Because $v^{ij} = v^{ji}$, we have:

$$\begin{aligned} u^i v^{jk} &= \frac{1}{3}\left(u^i v^{jk} + u^j v^{ki} + u^k v^{ij}\right) + \frac{1}{3}\left(2u^i v^{jk} - u^j v^{ki} - u^k v^{ij}\right) \\ &= \frac{1}{3}\left(u^i v^{jk} + u^j v^{ki} + u^k v^{ij}\right) + \frac{1}{3}(\delta^i_m \delta^j_n - \delta^i_n \delta^j_m) u^m v^{nk} \\ &\quad + \frac{1}{3}(\delta^i_m \delta^k_n - \delta^i_n \delta^k_m) u^m v^{nj} \\ &= \frac{1}{3}\left(u^i v^{jk} + u^j v^{ki} + u^k v^{ij}\right) + \frac{1}{3}\epsilon^{ijl}\epsilon_{mnl} u^m v^{nk} + \frac{1}{3}\epsilon^{ikl}\epsilon_{mnl} u^m v^{nj} \end{aligned}$$

i.e.,

$$(1,0) \;\otimes\; (2,0) \;=\; (3,0) \;\oplus\; (1,1)$$

With Young tableau technique, that is,

$$\begin{array}{|c|}\hline \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline & 1 & 1 \\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$$

2. In adjoint representation,

$$|u\rangle = u^i_j \left|{}^j_i\right\rangle = u^i_j \left|{}^j\right\rangle \otimes \left|{}_i\right\rangle$$

with $u^i_i = 0$. Equivalently,

$$|u\rangle = \left(u^i_j - \frac{1}{3}\delta^i_j u^l_l\right) \left|{}^j_i\right\rangle$$

In this manner we express the 8 independent basis vectors of the adjoint representation in the 9-dimensional tensor product space of $\mathbf{3} \otimes \bar{\mathbf{3}}$.

We know that the $SU(3)$ generators for tensor product space of $\mathbf{3} \otimes \bar{\mathbf{3}}$ are,

$$X_a^{3\times\bar{3}} = X_a^3 \otimes 1 + 1 \otimes X_a^{\bar{3}} = \frac{1}{2}\left[\lambda_a^3 \otimes 1 + 1 \otimes \lambda_a^{\bar{3}}\right]$$

where λ_a^3 and $\lambda_a^{\bar{3}}$ are representation matrices of Gell-Mann matrices λ_a in $\mathbf{3}$ and $\bar{\mathbf{3}}$, respectively.

Therefore, in adjoint representation,

$$\begin{aligned}
\langle u|X_a|v\rangle &= \left(u^i_j - \frac{1}{3}\delta^i_j u^m_m\right)\left(v^k_l - \frac{1}{3}\delta^k_l v^n_n\right)\left\langle {}^j_i\right| X_a \left|{}^l_k\right\rangle \\
&= \frac{1}{2}\left(u^i_j - \frac{1}{3}\delta^i_j u^m_m\right)\left(v^k_l - \frac{1}{3}\delta^k_l v^n_n\right)\left\langle {}^j\right| \otimes \left\langle {}_i\right| \left[\lambda^3_a \otimes 1\right. \\
&\qquad\qquad \left. +1 \otimes \lambda^{\bar{3}}_a\right] \left|{}^l\right\rangle \otimes \left|{}_k\right\rangle \\
&= \frac{1}{2}\left(u^i_j - \frac{1}{3}\delta^i_j u^m_m\right)\left(v^k_l - \frac{1}{3}\delta^k_l v^n_n\right)\left[\left\langle {}^j\right| \lambda^3_a \left|{}^l\right\rangle \delta_{ik}\right. \\
&\qquad\qquad \left. +\delta^{jl} \left\langle {}_i\right| \lambda^{\bar{3}}_a \left|{}_k\right\rangle\right]
\end{aligned}$$

3. The irrreducible Rep.**6** of $SU(3)$ is just Rep.(2,0) with highest weight $\vec{M} = 2\vec{M}_1$. The whole set of weights reads,

$$\begin{array}{l}
\vec{M} \\
\vec{M} - \vec{\alpha}_1 \\
\vec{M} - 2\vec{\alpha}_1 \\
\vec{M} - \vec{\alpha}_1 - \vec{\alpha}_2 \\
\vec{M} - 2\vec{\alpha}_1 - \vec{\alpha}_2 \\
\vec{M} - 2\vec{\alpha}_1 - 2\vec{\alpha}_2
\end{array}$$

In the 9-dimensional representation space of $\mathbf{3} \otimes \mathbf{3}$, the basis state of **6** with highest weight can be expressed as:

$$|\vec{M}\rangle = |\vec{M}_1\rangle \otimes |\vec{M}_1\rangle = |{}_1\rangle \otimes |{}_1\rangle$$

Recall that in the fundamental representation **3**,

$$|{}_1\rangle = |\vec{M}_1\rangle\,, \qquad |{}_2\rangle = 2E_{-\alpha_2}E_{-\alpha_1}|\vec{M}_1\rangle\,, \qquad |{}_3\rangle = \sqrt{2}E_{-\alpha_1}|\vec{M}_1\rangle\,.$$

we have in Rep.**6**,

$$\begin{aligned}
|\vec{M} - \vec{\alpha}_1\rangle &= E_{-\alpha_1}|\vec{M}\rangle \\
&= \left[E_{-\alpha_1} \otimes 1 + 1 \otimes E_{-\alpha_1}\right]|{}_1\rangle \otimes |{}_1\rangle \\
&= \frac{1}{\sqrt{2}}\left[|{}_3\rangle \otimes |{}_1\rangle + |{}_1\rangle \otimes |{}_3\rangle\right]
\end{aligned}$$

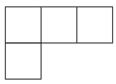
Similarly,

$$
\begin{aligned}
|\vec{M} - 2\vec{\alpha}_1\rangle &= |_3\rangle \otimes |_3\rangle \\
|\vec{M} - \vec{\alpha}_1 - \vec{\alpha}_2\rangle &= \frac{1}{2}\left[|_2\rangle \otimes |_1\rangle + |_1\rangle \otimes |_2\rangle\right] \\
|\vec{M} - 2\vec{\alpha}_1 - \vec{\alpha}_2\rangle &= \frac{1}{\sqrt{2}}\left[|_2\rangle \otimes |_3\rangle + |_3\rangle \otimes |_2\rangle\right] \\
|\vec{M} - 2\vec{\alpha}_1 - 2\vec{\alpha}_2\rangle &= |_2\rangle \otimes |_2\rangle
\end{aligned}
$$

The corresponding tensor components of these basis states are as follows:

$$
\begin{aligned}
(v_{\vec{M}})^{ij} &= \delta^{i1}\delta^{j1} \\
\left(v_{\vec{M}-\vec{\alpha}_1}\right)^{ij} &= \frac{1}{\sqrt{2}}\left[\delta^{i3}\delta^{j1} + \delta^{i1}\delta^{j3}\right] \\
\left(v_{\vec{M}-2\vec{\alpha}_1}\right)^{ij} &= \delta^{i3}\delta^{j3} \\
\left(v_{\vec{M}-\vec{\alpha}_1-\vec{\alpha}_2}\right)^{ij} &= \frac{1}{2}\left[\delta^{i2}\delta^{j1} + \delta^{i1}\delta^{j2}\right] \\
\left(v_{\vec{M}-2\vec{\alpha}_1-\vec{\alpha}_2}\right)^{ij} &= \frac{1}{\sqrt{2}}\left[\delta^{i2}\delta^{j3} + \delta^{i3}\delta^{j2}\right] \\
\left(v_{\vec{M}-2\vec{\alpha}_1-2\vec{\alpha}_2}\right)^{ij} &= \delta^{i2}\delta^{j2}
\end{aligned}
$$

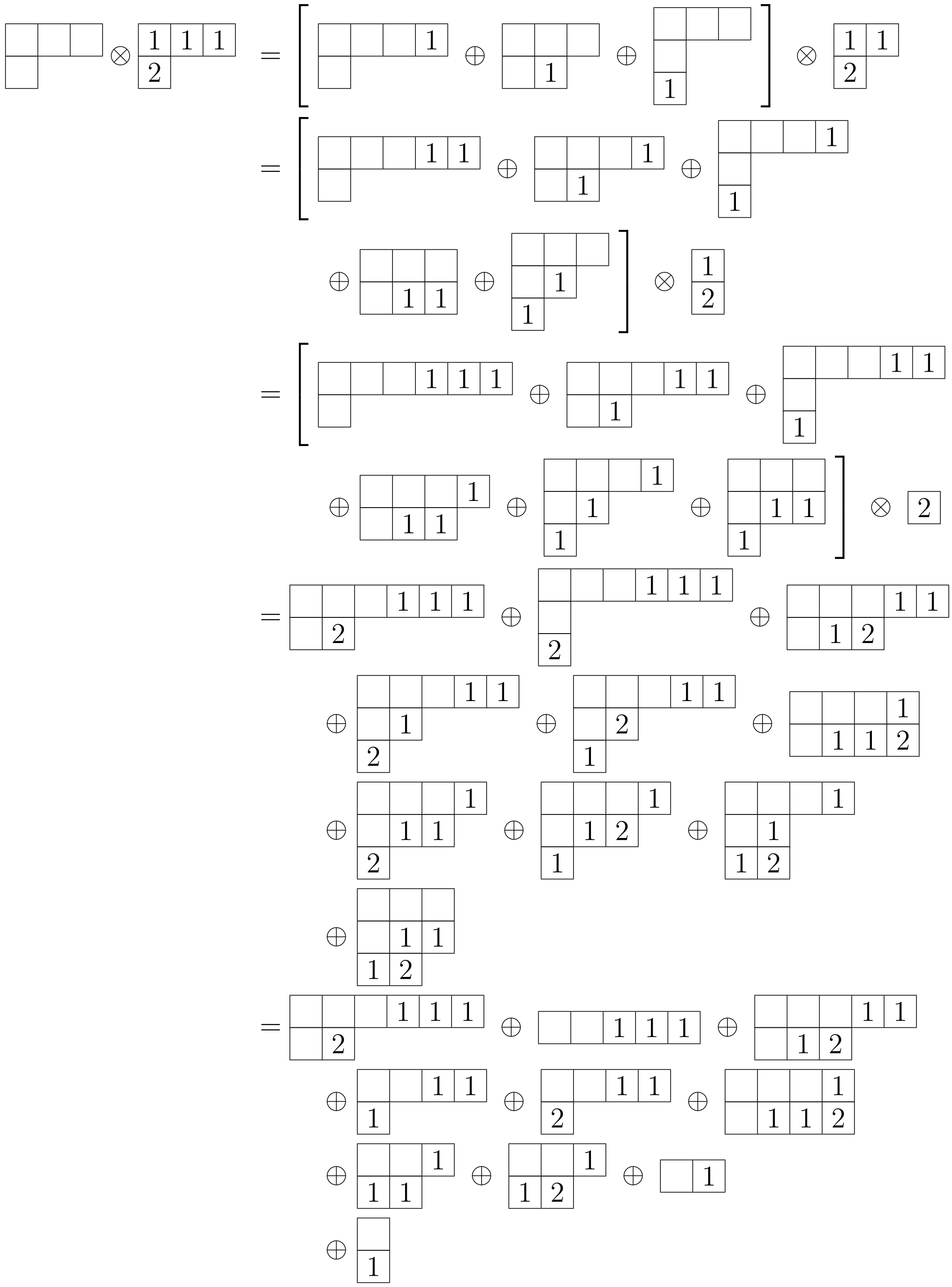
4. The tensor in irreducible Rep.$(2,1)$ of $SU(3)$ can be expressed as Young tableau



whose dimension is,

$$D_{(2,1)} = \frac{1}{2}(2+1)(1+1)(2+1+2) = 15.$$

Using Young tableau technique, we have:



Equivalently,

$$(2,1) \otimes (2,1) = (4,2) \oplus (5,0) \oplus (2,3) \oplus 2(3,1) \oplus (0,4) \oplus 2(1,2) \oplus (2,0) \oplus (0,1)$$

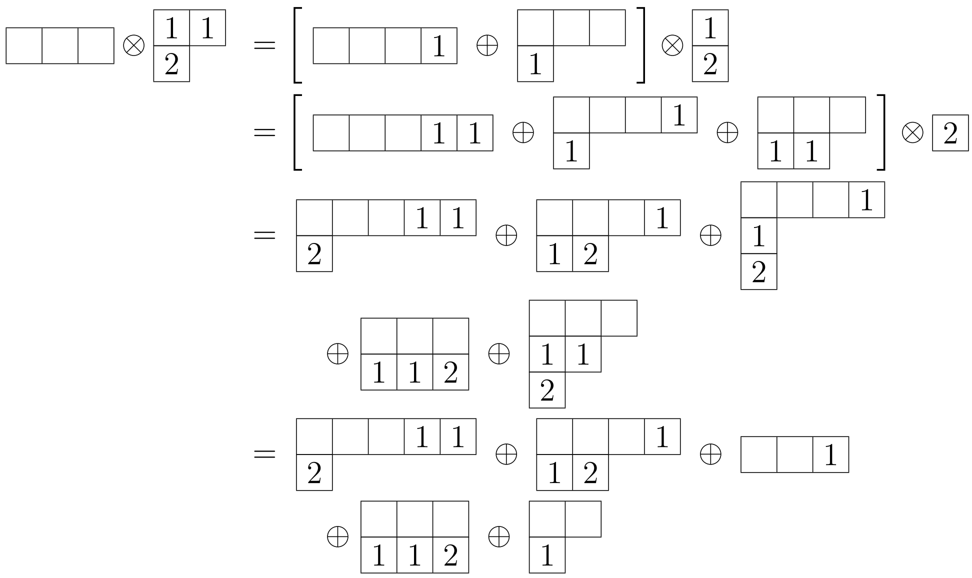
Among these irreducible representations, $(5,0)$ and $(2,0)$ are symmetric while $(0,1)$ is anti-symmetric. The dimensions of the involved irreducible representations in the decomposition are:

$$D_{(4,2)} = 60, \quad D_{(5,0)} = 21, \quad D_{(2,3)} = 42, \quad D_{(3,1)} = 24,$$
$$D_{(0,4)} = 15, \quad D_{(1,2)} = 15, \quad D_{(2,0)} = 6, \quad D_{(0,1)} = 3.$$

The dimensions on the two sides of the decomposition equation match each other,

$$15 \times 15 = 60 + 21 + 42 + 2 \times 24 + 15 + 2 \times 15 + 6 + 3$$

5. The irreducible representations **8** and **10** of $SU(3)$ are $(1,1)$ and $(3,0)$, respectively. With Young tableaux, we can recast $\mathbf{10} \otimes \mathbf{8}$ as,



i.e.,

$$(3,0) \otimes (1,1) = (4,1) \oplus (2,2) \oplus (3,0) \oplus (0,3) \oplus (1,1)$$

However, the dimensions on two sides of above equation do not match,

$$10 \times 8 \neq 35 + 27 + 10 + 10 + 8$$

Because $\mathbf{10} \otimes \mathbf{8}$ is equivalent to $\mathbf{8} \otimes \mathbf{10}$, we see that the irreducible Rep.$(0, 3)$ does not appear in the decomposition. This is because the Young tableau

		1
	1	1

is not valid. Therefore, the correct decomposition rule should be,

$$(3, 0) \otimes (1, 1) = (4, 1) \ \oplus \ (2, 2) \ \oplus \ (3, 0) \ \oplus \ (1, 1)$$

6. The general irreducible representation of $SU(3)$ is Rep.(n, m), with tensor expressed by Young tableau

The numbers of boxes in the first row and the second row are $(n + m)$ and m, respectively. In particular, the Rep.$(1, 1)$

is the adjoint representation of $SU(3)$. Consider the tensor product of Reps. $(1, 1)$ and (n, m):

$\otimes$

1	1	$\cdots$	1	1	1	$\cdots$	1
2	2	$\cdots$	2				

In its decomposition, there is a term given by Young tableau,

		1	$\cdots$	1	1	1	$\cdots$	1
	2	2	$\cdots$	2				
1								

which is equivalent to:

	1	$\cdots$	1	1	1	$\cdots$	1
2	2	$\cdots$	2				

This is very the irreducible Rep.(n, m) of $SU(3)$.

Chapter 5

Problems:

1. Show that the $su(N)$ algebra has an $su(N-1)$ subalgebra. How do the fundamental Rep.[1] of $SU(N)$ decompose into $SU(N-1)$ representations ?

2. Find $[3] \otimes [1]$ in $SU(5)$. Check that the dimensions work out.

3. Find $[3,1] \otimes [2,1]$ in $SU(6)$.

4. Find $[2] \otimes [1,1]$ in $SU(N)$, using the factors over hooks rule to check that the dimensions work out for arbitrary N.

Solutions:

1. In defining representation of $SU(N)$, its generators can be chosen as the N^2 traceless $N \times N$ matrices A_{ij} defined by,

$$(A_{ij})_{kl} = \delta_{ik}\delta_{jl} - \frac{1}{N}\delta_{ij}\delta_{kl}$$

where $i, j, k, l = 1, 2, \cdots, N$. Due to the fact

$$\sum_{i=1}^{N}(A_{ii})_{kl} = 0$$

the number of independent generators is $(N^2 - 1)$ rather than N^2, as expected. The Lie brackets between two generators are,

$$\begin{aligned}
[A_{ij},\ A_{kl}]_{mn} &= (A_{ij})_{ma}(A_{kl})_{an} - (A_{kl})_{ma}(A_{ij})_{an} \\
&= \left[\delta_{im}\delta_{ja} - \frac{1}{N}\delta_{ij}\delta_{ma}\right]\left[\delta_{ka}\delta_{ln} - \frac{1}{N}\delta_{kl}\delta_{an}\right] \\
&\quad - \left[\delta_{km}\delta_{la} - \frac{1}{N}\delta_{kl}\delta_{ma}\right]\left[\delta_{ia}\delta_{jn} - \frac{1}{N}\delta_{ij}\delta_{an}\right] \\
&= \delta_{im}\left[\delta_{jk}\delta_{ln} - \frac{1}{N}\delta_{kl}\delta_{jn}\right] - \frac{1}{N}\delta_{ij}\delta_{mk}\delta_{ln} + \frac{1}{N^2}\delta_{ij}\delta_{kl}\delta_{mn} \\
&\quad -\delta_{km}\left[\delta_{li}\delta_{jn} - \frac{1}{N}\delta_{ij}\delta_{ln}\right] + \frac{1}{N}\delta_{kl}\delta_{im}\delta_{jn} - \frac{1}{N^2}\delta_{kl}\delta_{ij}\delta_{mn} \\
&= \delta_{jk}\delta_{im}\delta_{ln} - \delta_{il}\delta_{km}\delta_{jn} \\
&= \delta_{jk}\left[\delta_{im}\delta_{ln} - \frac{1}{N}\delta_{il}\delta_{mn}\right] - \delta_{il}\left[\delta_{km}\delta_{jn} - \frac{1}{N}\delta_{kj}\delta_{mn}\right] \\
&= \delta_{jk}(A_{il})_{mn} - \delta_{il}(A_{kj})_{mn}
\end{aligned}$$

Namely,

$$[A_{ij},\ A_{kl}] = \delta_{jk}A_{il} - \delta_{il}A_{kj}$$

where $i, j, k, l = 1, 2, \cdots, N$. Obviously, if we restrict the values of i, k, k, l to the region $i, j, k, l = 1, 2, \cdots, N-1$, the above commutators hold also. The number of generators of $SU(N)$ in this subset is naively $(N-1)^2$. However, due to the fact

$$\sum_{i=1}^{N-1}(A_{ii})_{mn} = \sum_{i=1}^{N-1}\left[\delta_{im}\delta_{in} - \frac{1}{N}\delta_{ii}\delta_{mn}\right] = \sum_{i=1}^{N-1}\delta_{im}\delta_{in} - \left(\frac{N-1}{N}\right)\delta_{mn}$$

the number of independent generators in this subset is $[(N-1)^2 - 1]$. Consequently, the generators in this subset generate a subgroup $SU(N-1)$ of $SU(N)$.

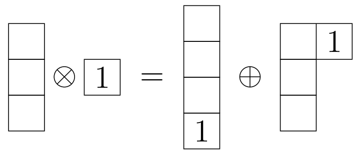
The fundamental Rep.[1] of $SU(N)$ decompose into the following $SU(N-1) \otimes U(1)$ representations:

$$\square = \left(\square \quad \bullet\right) \oplus \left(\bullet \quad \square\right)$$

i.e.,

$$\boldsymbol{N} = (\boldsymbol{N-1}) \oplus \boldsymbol{1}$$

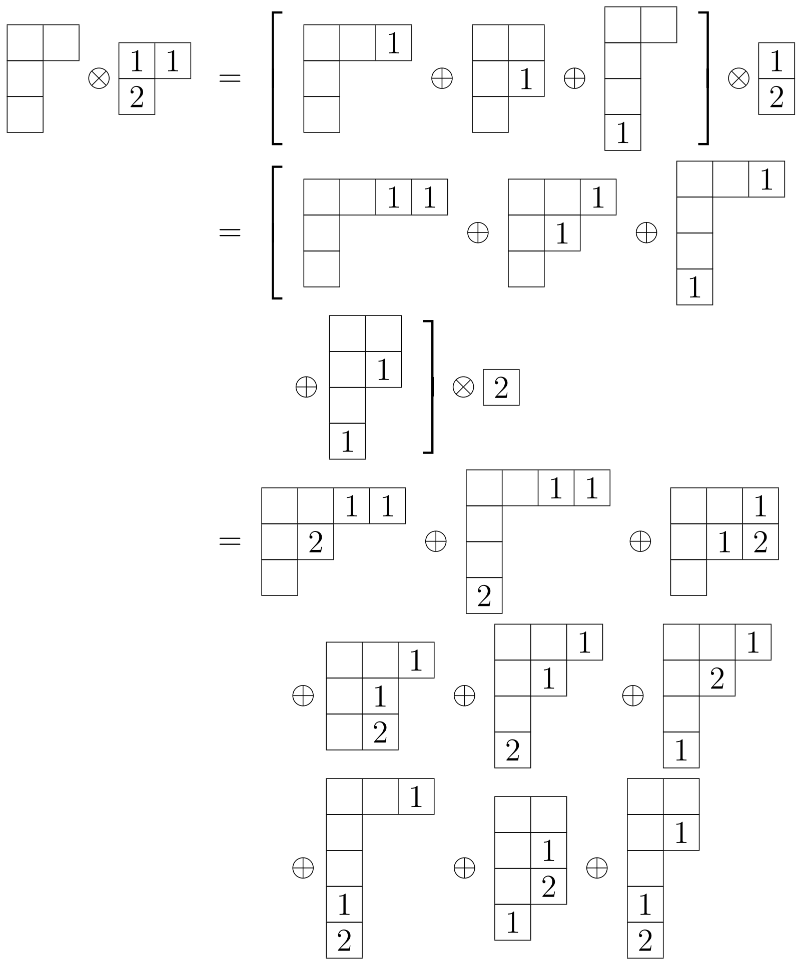
2. Using Young tableau technique, the decomposition of $[3] \otimes [1]$ in $SU(5)$ reads,



The dimensions on two sides match:

$$10 \times 5 = 5 + 45$$

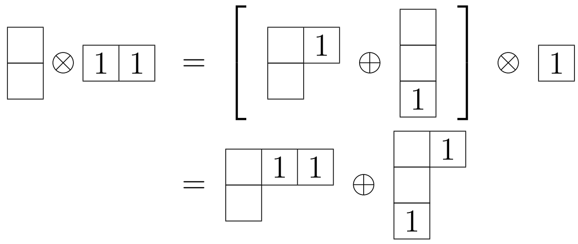
3. The decomposition of $[3, 1] \otimes [2, 1]$ of $SU(6)$ is,



The dimensions on two sides of this equation match:

$$105 \times 70 = 2520 + 720 + 1176 + 840 + 840 + 840 + 120 + 210 + 84$$

4. The decomposition of $[2] \otimes [1,1]$ of $SU(N)$ reads,



The dimensions on two sides of this equation are

$$\frac{N(N-1)}{2} \times \frac{N(N+1)}{2} = \frac{1}{4}N^2(N^2-1)$$

and

$$\frac{N(N+1)(N+2)(N-1)}{8} + \frac{N(N+1)(N-1)(N-2)}{8} = \frac{1}{4}N^2(N^2-1)$$

They match each other, as expected.