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数理方程B

练习簿

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1.1 (1) 在极坐标系下，求方程 $\Delta_2 u=0$ 的形如 $u=u(r)$ $(r=\sqrt{x^2+y^2}\neq 0)$ 的解

(2) 在球坐标系下，求方程 $\Delta_3 u+k^2 u=0$ (k 为正常数) 的形如 $u=u(r)$ 的解.

解：(1) $x=r\cos\theta$，$y=r\sin\theta$ $\quad u=u(x,y)$

$\Delta_2 u=u_{xx}+u_{yy}$

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta$$

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}=-\frac{\partial u}{\partial x}r\sin\theta+\frac{\partial u}{\partial y}r\cos\theta.$$

$$\cancel{\frac{\partial^2 u}{\partial r^2}=\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial r}\cos\theta+\frac{\partial^2 u}{\partial y^2}\cdot\frac{\partial y}{\partial r}\sin}$$

$$\frac{\partial^2 u}{\partial r^2}=\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial r}\cos\theta+\frac{\partial^2 u}{\partial x\partial y}\cdot\frac{\partial y}{\partial r}\cos\theta+\frac{\partial^2 u}{\partial y\partial x}\cdot\frac{\partial x}{\partial r}\sin\theta+\frac{\partial^2 u}{\partial y^2}\cdot\frac{\partial y}{\partial r}\sin\theta$$

$$=\frac{\partial^2 u}{\partial x^2}\cos^2\theta+2\frac{\partial^2 u}{\partial x\partial y}\sin\theta\cos\theta+\frac{\partial^2 u}{\partial y^2}\sin^2\theta$$

$$\frac{\partial^2 u}{\partial \theta^2}=-\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial \theta}r\sin\theta-\frac{\partial^2 u}{\partial x\partial y}\cdot\frac{\partial y}{\partial \theta}r\sin\theta-\frac{\partial u}{\partial x}r\cos\theta$$

$$+\frac{\partial^2 u}{\partial y\partial x}\cdot\frac{\partial x}{\partial \theta}r\cos\theta+\frac{\partial^2 u}{\partial y^2}\cdot\frac{\partial y}{\partial \theta}r\cos\theta-\frac{\partial u}{\partial y}r\sin\theta$$

$$=\frac{\partial^2 u}{\partial x^2}r^2\sin^2\theta+\frac{\partial^2 u}{\partial y^2}r^2\cos^2\theta-2\frac{\partial^2 u}{\partial x\partial y}r^2\cos\theta\sin\theta-\frac{\partial u}{\partial x}r\cos\theta-\frac{\partial u}{\partial y}r\sin\theta.$$

$$\therefore\ \Delta_2 u=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$\because u=u(r)$ 与 θ 无关 $\quad\therefore \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0$

$\therefore \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=0 \qquad$ 令 $v=\frac{\partial u}{\partial r}$ 则 $\frac{\partial v}{\partial r}+\frac{1}{r}v=0$

$\therefore \frac{dv}{dr}=-\frac{1}{r}v \qquad \therefore \int\frac{1}{v}dv=-\int\frac{1}{r}dr$

$\therefore \ln v=-\ln r+\ln C \quad\Rightarrow\quad v=\frac{C}{r}$. C 常数

$\therefore \frac{\partial u}{\partial r}=\frac{C}{r} \qquad \therefore u=C_1+C_2\ln r$. C_1. C_2 为任意常数.

(2) 球坐标系下，
$$\begin{cases} x=r\sin\theta\cos\varphi \\ y=r\sin\theta\sin\varphi \\ z=r\cos\theta\end{cases}\qquad u=u(x,y,z)$$

$\Delta_3 u$ 在球坐标系下的形式为

$$\Delta_3 u=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}$$

$\because u=u(r)$

$\therefore \Delta_3 u+k^2u=0$ 即

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+k^2u=0$$

$$\therefore \left(2r\cdot\frac{\partial u}{\partial r}+r^2\frac{\partial^2 u}{\partial r^2}\right)+k^2r^2u=0$$

$$\therefore \frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+k^2u=0$$

$$\frac{d^2u}{dr^2}+\frac{2}{r}\frac{du}{dr}+k^2u=0$$

$$\boxed{r^2u_{rr}+2ru_r+k^2r^2u=0}$$

不是欧拉方程

$u=u(r)$，设 $u=R(r)$

即 $r\cdot R''+2R'+k^2\cdot rR=0$ ，$k>0$

令 $y=r\cdot R$

$\therefore y''+k^2y=0$ ，$k>0$

$\therefore y=C_1\cos k r+C_2\sin kr$

$\therefore R(r)=\frac{C_1}{r}\cos kr+\frac{C_2}{r}\sin kr$

① $k=0$ 时，$r^2u_{rr}+2ru_r=0$ 即 $r u_{rr}+2u_r=0$

令 $u_r=v(r)$ 则 $rv'+2v=0$

$$r\cdot\frac{dv}{dr}=-2v \Rightarrow \frac{dv}{v}=-2\frac{dr}{r}\Rightarrow\int\frac{dv}{v}=-2\int\frac{dr}{r}$$

$$\therefore \ln v=\ln C'-2\ln r \Rightarrow v=\frac{C'}{r^2}\Rightarrow\frac{du}{dr}=\frac{C'}{r^2}$$

$$\therefore \int du=C'\int\frac{1}{r^2}dr \Rightarrow u=\frac{C}{r}$$ （C 为任意常数）.

② $k\neq0$ 时，猜一特解为 $u_1=\frac{e^{ikr}}{r}$

$$u_2=u_1\int\frac{1}{u_1^2}e^{-\int\frac{2}{r}dr}dr\propto\frac{e^{-ikr}}{r}$$

$$\therefore u=\frac{1}{r}\left(C_1e^{ikr}+C_2e^{-ikr}\right).$$

6. 设 $u=u(x,y,z)$. 求下列方程的通解：

(1) $\frac{\partial u}{\partial y}+a(x,y)u=0$

(2) $u_{xy}+u_y=0$

(3) $u_{tt}=a^2u_{xx}+3x^2$ （设 $u=u(x,t)$）.

解：(1) 乘上 $e^{\int a(x,y)dy}$. 得到：

$$e^{\int a(x,y)dy}u_y+e^{\int a(x,y)dy}a(x,y)u=0$$

$$\therefore \left(e^{\int a(x,y)dy}u\right)_y=0$$

$$\therefore e^{\int a(x,y)dy}u=f(x,z)$$

$$\therefore u=e^{-\int a(x,y)dy}f(x,z)$$

(2) $(u_x+u)_y=0$ $\therefore u_x+u=f'(x,z)$.

乘上 $e^{\int 1dx}$ $\Rightarrow e^xu_x+e^xu=f(x,z)$

$\Rightarrow (e^xu)_x=f(x,z)$

$$\therefore e^xu=\int f(x,z)dx+g(y,z)$$

$$\therefore u=e^{-x}\left[\int f(x,z)dx+g(y,z)\right]$$

(3) ① $a=0$ 时, $u_{tt}=3x^2$

$$\Rightarrow u_t=\int 3x^2dt+f(x)=3x^2t+f(x)$$

$$\Rightarrow u=\int 3x^2t\,dt+\int f(x)dt+h(x)$$

$$=\frac{3}{2}x^2t^2+tf(x)+h(x).$$

② $a\neq 0$ 时, 设一个特解为 $v=Ax^4$ 则 $0=a^2\cdot 12Ax^2+3x^2$

$$\Rightarrow A=-\frac{1}{4a^2}\Rightarrow v=-\frac{1}{4a^2}x^4.$$

设 $w=u-v$ ~~则 $u_{tt}=a^2u_{xx}$~~ 则 $w_{tt}=a^2w_{xx}$

$$\Rightarrow w=f(x+at)+g(x-at)$$

$$\therefore u=f(x+at)+g(x-at)-\frac{1}{4a^2}x^4.$$

7. 设有一根具有绝热的侧表面的均匀细杆，它的初始温度为 $\varphi(x)$，两端满足下列边界条件之一：

(1) 一端 $(x=0)$ 绝热，另一端 $(x=l)$ 保持常温 u_0。

(2) 两端分别有恒定的热流密度 q_1 及 q_2 进入。

(3) 一端 $(x=0)$ 温度为 $\mu(t)$，另一端 $(x=l)$ 与温度为 $\theta(t)$ 的介质有热交换。

试分别写出上述三种热过程的定解问题。

解：(1)
$$\begin{cases} u_t = a^2 u_{xx}, & 0<x<l,\ t>0. \\ u(0,x)=\varphi(x) \\ u_x(t,0)=0,\ u(t,l)=u_0 \end{cases}$$

(2)
$$\begin{cases} u_t = a^2 u_{xx}, & 0<x<l,\ t>0 \\ u(0,x)=\varphi(x) \\ u_x(t,0) = -\frac{q_1}{k},\ u_x(t,l) = \frac{q_2}{k} \end{cases}$$

(3)
$$u_t = a^2 u_{xx},\ 0<x<l,\ t>0$$
$$u(0,x)=\varphi(x)$$
$$u(0,t)=\mu(t),\ \cancel{h[u(t,l)-\theta(t)] = k u_x(t,l)}$$
$$\cancel{h[u(t,l)-\theta(t)] = -k u_x(t,l)}$$

(3)
$$\begin{cases} u_t = a^2 u_{xx},\ 0<x<l,\ t>0 \\ u(0,x)=\varphi(x) \\ h[\theta(t)-u(t,l)] = k u_x(t,l) \\ u(0,t)=\mu(t) \end{cases}$$
h 为热交换系数。

9. 求下列定解问题的解：

(1) $u_t = x^2$, $u(0,x) = x^2$.

(4) 古尔萨(Goursat)问题.

$$\begin{cases} u_{tt} = u_{xx} \\ u|_{t+x=0} = \varphi(x), \ \varphi(0) = \psi(0) \\ u|_{t-x=0} = \psi(x) \end{cases}$$

解：(1) $\because u_t = x^2$ $\therefore \frac{\partial u}{\partial t} = x^2 \Rightarrow u = t\cdot x^2 + f(x)$

$\because u(0,x) = t\cdot x^2 + f(x) \Rightarrow f(x) = x^2$

$\therefore u(t,x) = t\cdot x^2 + x^2$.

~~⇒~~ (4) 自由弦振动的通解为：

$u(t,x) = h_1(x+at) + h_2(x-at)$ $\because a=1$

$\therefore u(t,x) = h_1(x+t) + h_2(x-t)$

$u|_{t=-x} = \varphi(x) = h_1(0) + h_2(2x)$ $u|_{t=x} = \psi(x) = h_1(2x) + h_2(0)$

$\Rightarrow h_1(x) = \psi(\frac{x}{2}) - h_2(0)$ $h_1(0) = \psi(0) - h_2(0)$ ~~$\Rightarrow h_1(0) + h_2(0) = \psi(0) = \varphi(0)$~~

$h_2(x) = \varphi(\frac{x}{2}) - h_1(0)$ $h_2(0) = \varphi(0) - h_1(0)$

$\therefore u(t,x) = h_1(x+t) + h_2(x-t)$

$= \psi(\frac{x+t}{2}) - h_2(0) + \varphi(\frac{x+t}{2}) - h_1(0)$

$= \psi(\frac{x+t}{2}) + \varphi(\frac{x+t}{2}) - \varphi(0)$

10. 利用叠加原理和齐次化原理求解

$$\begin{cases}\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}=f(t,x) & (t>0,\ -\infty<x<+\infty)\\ u(0,x)=\varphi(x) & (a\neq 0,\ a\text{为常数}).\end{cases}$$

解：利用叠加原理把方程化为两个解相加.

① $\begin{cases}\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}=0\\ u(0,x)=\varphi(x)\end{cases}$ ② $\begin{cases}\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}=f(t,x)\\ u(0,x)=0\end{cases}$

先解第一个方程，令 $\xi=x-at$，$\eta=t$

$\hat{u}(\eta,\xi)=u(x(\eta,\xi),t(\eta,\xi))$

$u(x,t)=\tilde{u}(\eta(x,t),\xi(x,t))=\tilde{u}(x-at,t)$

$\therefore \frac{\partial u}{\partial t}=\frac{\partial\tilde{u}}{\partial\eta}-a\cdot\frac{\partial\tilde{u}}{\partial\xi}$ $\frac{\partial u}{\partial x}=\frac{\partial\tilde{u}}{\partial\xi}$

$\therefore \frac{\partial\tilde{u}}{\partial\eta}=\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}=0$ $\therefore \tilde{u}=G(\xi)\Rightarrow u(t,x)=G(x-at)$

由 $u|_{t=0}=\varphi(x)\Rightarrow G(x)=\varphi(x)$

$\Rightarrow u(t,x)=\varphi(x-at)$

利用齐次化原理求解②：

考虑方程 $\begin{cases}\frac{\partial w}{\partial t}+a\frac{\partial w}{\partial x}=0 & (t\geq\tau)\\ w(\tau,x)=f(\tau,x)\end{cases}$

令 $t'=t-\tau$ 则方程化为 $\begin{cases}\frac{\partial w}{\partial t'}+a\frac{\partial w}{\partial x}=0 & (t'\geq 0)\\ w|_{t'=0}(x)=f(\tau,x)\end{cases}$

由上述齐次方程的解可知

$w(t',x)=f(\tau,x-at')$

$\Rightarrow w(t,x)=f(\tau,x-a(t-\tau))$

$\therefore u(t,x) = \int_0^t f(\tau, x-a(t-\tau))\,d\tau$

$\therefore$ 原方程的解为

$$u(t,x) = \varphi(x-at) + \int_0^t f(\tau, x-a(t-\tau))\,d\tau.$$

3.12. 2. 5(2)(6)(7)

2、解下列固有值问题：

(1) $\begin{cases} y''-2ay'+\lambda y=0 & (0<x<1, a为常数) \\ y(0)=y(1)=0 \end{cases}$

(2) $\begin{cases} (r^2R')'+\lambda r^2R=0 & (0<r<a) \\ |R(0)|<+\infty, R(a)=0 & (提示：令y=rR) \end{cases}$

(3) $\begin{cases} y^{(4)}+\lambda y=0 & (0<x<l) \\ y(0)=y(l)=y''(0)=y''(l)=0. \end{cases}$

解：(1) 此微分方程的特征方程是

$$r^2-2ar+\lambda=0, \quad \Delta=4a^2-4\lambda$$

$\Delta\geq 0$ 时，$r_{1,2}=\dfrac{2a\pm\sqrt{4a^2-4\lambda}}{2}=a\pm\sqrt{a^2-\lambda}$

$\Rightarrow y=C_1e^{(a+\sqrt{a^2-\lambda})x}+C_2e^{(a-\sqrt{a^2-\lambda})x}$

$\because y(0)=y(1)=0$

$\therefore \begin{cases} C_1+C_2=0 \\ C_1e^{a+\sqrt{a^2-\lambda}}+C_2e^{a-\sqrt{a^2-\lambda}}=0 \end{cases} \Rightarrow \begin{cases} C_1=0 \\ C_2=0 \end{cases}, y=0$

$\Delta<0$ 时，$y=C_1e^{ax}\sin(\sqrt{\lambda-a^2})x+C_2e^{ax}\cos(\sqrt{\lambda-a^2})x$

$\because y(0)=y(1)=0$

$\therefore \begin{cases} C_2=0 \\ C_1e^a\sin(\sqrt{\lambda-a^2})=0 \end{cases} \Rightarrow \begin{cases} C_2=0 \\ \sqrt{\lambda-a^2}=k\pi, k\in Z. \end{cases}$

$\Rightarrow y=C_1e^{ax}\sin k\pi x.$

$\therefore \lambda_n=a^2+(n\pi)^2, \quad y_n(x)=e^{ax}\sin n\pi x \quad (n=1,2,\cdots)$

3.12.　2.　5(2)(6)(7)

2. 解固有值问题

(1) $\begin{cases} y''-2ay'+\lambda y=0 & (0<x<1,\ a为常数) \\ y(0)=y(1)=0 \end{cases}$

解：对照SL：$[k(x)X']'-q(x)X+\lambda\rho(x)X=0$

$q(x)=0$

由SL理论，固有值大于0.

∵特征方程为 $r^2-2ar+\lambda=0$

$\Rightarrow r_1=a+\sqrt{a^2-\lambda}$　　$r_2=a-\sqrt{a^2-\lambda}$

$\lambda<a$. 舍去

$\lambda=a$. 舍去.

$\lambda>a$，$y=C_1e^{ax}\cos(\sqrt{\lambda-a^2}x)+C_2e^{ax}\sin(\sqrt{\lambda-a^2}x)$.

代入边界条件 $y(0)=y(1)=0$

$\Rightarrow \begin{cases} C_1=0 \\ \sin\sqrt{\lambda-a^2}=0 \quad \sqrt{\lambda-a^2}=n\pi.\ n=1,2,\cdots \end{cases}$

∴ 固有值 $\lambda_n=a^2+(n\pi)^2$，　$y_n(x)=e^{ax}\sin n\pi x$，$n=1,2,\cdots$

(2) $\begin{cases} (r^2R')'+\lambda r^2R=0 & (0<r<a) \\ |R(0)|<+\infty,\ R(a)=0 \end{cases}$　　(提示：令 $y=rR(r)$)

解：$(r^2R')'+\lambda r^2R=0$

即 $2rR'+r^2R''+\lambda r^2R=0$，$2R'+rR''+\lambda rR=0$

令 $y=rR$ 则 $\begin{cases} y''+\lambda y=0, & y=y(r) \quad 0<r<a \\ y(0)=0,\ y(a)=0 \end{cases}$

$\because q(x)=0$，且满足Ⅰ类边界条件，$\therefore$由SL理论，固有值大于0.

$\therefore y_n(r)=C_1\cos\sqrt{\lambda}\,r+C_2\sin\sqrt{\lambda}\,r$

代入 $y(0)=y(a)=0$

$$\Rightarrow\begin{cases}C_1=0\\ \sin\sqrt{\lambda}a=0, \quad \sqrt{\lambda}a=n\pi, \quad n=1,2,\cdots\end{cases}$$

$\therefore \lambda_n=\left(\frac{n\pi}{a}\right)^2$，$y_n(r)=\sin\frac{n\pi r}{a}$ ，$n=1,2,\cdots$

即 $R_n(r)=\frac{1}{r}\sin\frac{n\pi r}{a}$ $n=1,2,\cdots$

(3) $\begin{cases}y^{(4)}+\lambda y=0 \quad (0<x<l)\\ y(0)=y(l)=y''(0)=y''(l)=0\end{cases}$

~~特征方程 $r^4+\lambda=0$~~

解：① $\lambda=0$时，$y=Ax^3+Bx^2+Cx+D$，代入边界条件有 $\begin{cases}D=0\\ Al^3+Bl^2+Cl+D=0\\ 2B=0\\ 6Al+2B=0\end{cases}$

$\Rightarrow A=B=C=D=0 \Rightarrow y=0$

② $\lambda>0$时，令$\lambda=\beta^4$ $\therefore$特征方程根为 $\begin{cases}s_1=\beta(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i)\\ s_2=\beta(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i)\\ s_3=\beta(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i)\\ s_4=\beta(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i)\end{cases}\xrightarrow{令\alpha=\frac{\sqrt{2}}{2}\beta}\begin{cases}s_1=\alpha+\alpha i\\ s_2=\alpha-\alpha i\\ s_3=-\alpha+\alpha i\\ s_4=-\alpha+\alpha i\end{cases}$

解为$y=e^{-\alpha x}(A\cos\alpha x+B\sin\alpha x)+e^{\alpha x}(C\cos\alpha x+D\sin\alpha x)$

代入初始条件

$$\Rightarrow\begin{cases}A+C=0\\ -A+B+C+D=0\\ (e^{-\alpha l}\cos\alpha l)A+(e^{-\alpha l}\sin\alpha l)B+(e^{\alpha l}\cos\alpha l)C+(e^{\alpha l}\sin\alpha l)D=0\\ -e^{-\alpha l}(\sin\alpha l+\cos\alpha l)A+e^{-\alpha l}(\cos\alpha l-\sin\alpha l)B+e^{\alpha l}(\cos\alpha l-\sin\alpha l)C+\\ \quad e^{\alpha l}(\sin\alpha l+\cos\alpha l)D=0\end{cases}$$

$\Rightarrow A=B=C=D=0 \Rightarrow y=0$

③ $\lambda<0$时，令$\lambda=-\beta^4$，特征方程根为$s=\pm\beta, \pm i\beta$

$$y=Ae^{\beta x}+Be^{-\beta x}+C\cos\beta x+D\sin\beta x$$

代入边界条件
$$\begin{cases} A+B+C=0 \\ Ae^{\beta l}+Be^{-\beta l}+C\cos\beta l+D\sin\beta l=0 \\ A+B-C=0 \\ Ae^{\beta l}+Be^{-\beta l}-C\cos\beta l-D\sin\beta l=0 \end{cases}$$

$\Rightarrow A=0 \quad B=0 \quad C=0$

$\sin\beta l=0 \Rightarrow \beta=\frac{n\pi}{l}, n=1,2,3\cdots$

固有值 $\lambda_n=-(\frac{n\pi}{l})^4$

固有函数 $y_n(x)=\sin\frac{n\pi x}{l}, n=1,2,3\cdots$

5. (2) 解定解问题

$$\begin{cases} u_t=a^2u_{xx} \quad (0<x<l, t>0) \\ u(t,0)=u(t,l)=0 \\ u(0,x)=x(l-x) \end{cases}$$

解：设$u(t,x)=T(t)X(x)$

$$\Rightarrow \frac{T'}{a^2T}=\frac{X''}{X}=-\lambda$$

可化为 $T'+\lambda a^2T=0$，$\begin{cases} X''+\lambda X=0 \\ X(0)=X(l)=0 \end{cases}$

分情况讨论：

① $\lambda<0$ 时，令 $\lambda=-k^2$，$X=Ae^{kx}+Be^{-kx}$

$\because X(0)=X(l)=0$ $\therefore \begin{cases} A+B=0 \\ Ae^{kl}+Be^{-kl}=0 \end{cases} \Rightarrow A=B=0. \Rightarrow X\equiv 0.$（舍）

② $\lambda=0$ 时，$X=Ax+B$

$\because X(0)=X(l)=0 \Rightarrow A=B=0 \Rightarrow X\equiv 0$（舍）

③ $\lambda=k^2>0$ 时，$X=A\cos kx+B\sin kx$

$\because X(0)=X(l)=0 \Rightarrow \begin{cases} A=0 \\ \sin kl=0 \end{cases} \Rightarrow kl=n\pi,\ n=1,2,\cdots$

固有值 $\lambda_n=\left(\frac{n\pi}{l}\right)^2$

固有函数 $X_n(x)=\sin\frac{n\pi x}{l}$，$n=1,2,\cdots$

固定 λ_n, $T_n'+\lambda_n a^2 T_n=0$ $\quad \frac{T_n'}{T_n}=-\lambda_n a^2$，$(\ln T_n)'=-\lambda_n a^2$，$\ln T_n=-\lambda_n a^2 t$

$\Rightarrow T_n(t)=e^{-\lambda_n a^2 t}=e^{-\left(\frac{n\pi a}{l}\right)^2 t}$，$n=1,2,\cdots$

利用叠加原理

$$u(t,x)=\sum_{n=1}^{+\infty}u_n(t,x)=\sum_{n=1}^{+\infty}C_n e^{-\left(\frac{n\pi a}{l}\right)^2 t}\sin\frac{n\pi x}{l}$$

代入初始条件

$$x(l-x)=\sum_{n=1}^{+\infty}C_n\sin\frac{n\pi x}{l}$$

展开：

$$C_n=\frac{\langle x(l-x),X_n\rangle}{\langle X_n,X_n\rangle}=\frac{\int_0^l x(l-x)X_n\,dx}{\int_0^l X_n^2\,dx}=\frac{\int_0^l x(l-x)\sin\frac{n\pi x}{l}\,dx}{\int_0^l \sin^2\left(\frac{n\pi x}{l}\right)dx}$$

$$=\frac{2}{l}\int_0^l x(l-x)\sin\frac{n\pi x}{l}\,dx=\frac{2}{l}\left(l\int_0^l x\sin\frac{n\pi x}{l}\,dx-\int_0^l x^2\sin\frac{n\pi x}{l}\,dx\right)$$

$$=\frac{2}{l}\left[l\left(-\frac{xl}{n\pi}\cos\frac{n\pi x}{l}+\frac{l^2}{(n\pi)^2}\sin\frac{n\pi}{l}x\right)\Big|_0^l-\left(-\frac{x^2 l}{n\pi}\cos\frac{n\pi x}{l}+\frac{2xl^2}{(n\pi)^2}\sin\frac{n\pi x}{l}+\frac{2l^3}{(n\pi)^3}\cos\frac{n\pi x}{l}\right)\Big|_0^l\right.$$

$$= \frac{4l^2}{(n\pi)^3}(1-\cos n\pi)$$

$$= \frac{4l^2}{(n\pi)^3}[1-(-1)^n]$$

∴定解问题的解为

$$u(t,x)=\sum_{n=1}^{+\infty}\frac{4l^2}{(n\pi)^3}[1-(-1)^n]e^{-(\frac{n\pi a}{l})^2 t}\sin\frac{n\pi x}{l}$$

∵ n为偶数时，$u(t,x)=0$ ∴ ~~令~~上式 n为奇数，

改写为：

$$u(t,x)=\sum_{n=0}^{+\infty}\frac{8l^2}{(2n+1)^3\pi^3}e^{-[\frac{(2n+1)\pi a}{l}]^2 t}\sin\frac{(2n+1)\pi x}{l}$$

(6) 环域内的狄氏问题

$$\begin{cases}\Delta_2 u=0 & (a<r<b)\\ u(a,\theta)=1,\ u(b,\theta)=0\end{cases}$$

先观察，发现 u与θ无关.

设 $u=u(r)$.

解：即 $\begin{cases}\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0\\ u(a,\theta)=1,\ u(b,\theta)=0\end{cases}$

设 $u(r,\theta)=R(r)\Theta(\theta)$ 代入方程得：

$$R''\Theta+\frac{1}{r}R'\Theta+\frac{1}{r^2}R\Theta''=0$$

$$(r^2R''+rR')\Theta=-R\Theta''$$

$$\Rightarrow 设\ \frac{r^2R''+rR'}{R}=-\frac{\Theta''}{\Theta}=-\lambda$$

$$r^2R''+rR'-\lambda R=0$$

$\Rightarrow$ 固有值问题 $\begin{cases}\Theta''+\lambda\Theta=0, & 0<\theta<2\pi\\ \Theta(0)=\Theta(2\pi)\end{cases}$

由S-L理论可知 $\lambda \geq 0$

① $\lambda_0=0$时，$\Theta_0(\theta)=1$

② $\lambda>0$时，令$\lambda=n^2$， $\Rightarrow$ $\Theta(\theta)=A_n\cos n\theta+B_n\sin n\theta$

$\lambda_n=n^2$，$\underline{\Theta_n(\theta)=\cancel{A_n}A\cos n\theta+B\sin n\theta}$ $n=1,2,\cdots$

解关于$R_n(\theta)$的微分方程

$$r^2R''+rR'-\lambda R=0$$

(1) $\lambda_0=0$时，$r^2R''+rR'=0$ $\Rightarrow \frac{(R')'}{R'}=-\frac{1}{r} \Rightarrow (\ln R')'=-\frac{1}{r} \Rightarrow \ln R'=\ln C-\ln r=\ln$

$\Rightarrow R'=\frac{C}{r}$ $\Rightarrow R=C_1\ln r+C_2$

即$R_0=C_{0,1}\ln r+C_{0,2}$

(2) $\lambda_n=n^2$时，$r^2R''+rR'-\lambda R=0$

令$r=e^t$ 得 $\frac{d^2R}{dt^2}-n^2R=0$

$\Rightarrow R_n=C_{n,1}e^{nt}+C_{n,2}e^{-nt}$

$\underline{=C_{n,1}r^n+C_{n,2}r^{-n}}$，$n=1,2,\cdots$

$\Rightarrow u(r,\theta)=\cancel{C_{0,1}\ln r+C_{0,2}}+\sum_{n=1}^{+\infty}(E_nr^n\cos n\theta+F_nr^{-n}\cos n\theta+G_nr^n\sin n\theta+H_nr^{-n}\sin n\theta)$

$C_{0,1}\ln r+C_{0,2}$

代入边界条件 $u(a,\theta)=1$ $u(b,\theta)=0$ 得

$$\begin{cases}1=C_{0,1}\ln a+C_{0,2}+\sum_{n=1}^{+\infty}(E_na^n\cos n\theta+F_na^{-n}\cos n\theta+G_na^n\sin n\theta+H_na^{-n}\sin n\theta)\\0=C_{0,1}\ln b+C_{0,2}+\sum_{n=1}^{+\infty}(E_nb^n\cos n\theta+F_nb^{-n}\cos n\theta+G_nb^n\sin n\theta+H_nb^{-n}\sin n\theta)\end{cases}$$

$\because$ $u(a,\theta)=1$，$u(b,\theta)=0$ 皆不是∞. $\therefore E_n=F_n=G_n=H_n=0$

$$\Rightarrow\begin{cases}C_{0,1}=\frac{1}{\ln a-\ln b}\\C_{0,2}=\frac{\ln b}{\ln b-\ln a}\end{cases}\Rightarrow u(r,\theta)=\frac{\ln r}{\ln a-\ln b}+\frac{\ln b}{\ln b-\ln a}$$

$$=\frac{\ln r-\ln b}{\ln a-\ln b}$$

(7) 扇形域内的狄氏问题

$$\begin{cases}\Delta_2 u=0 \quad (r<a,\ 0<\theta<\alpha) \\ u(r,0)=u(r,\alpha)=0 \\ u(a,\theta)=f(\theta)\end{cases}$$

解：即 $\begin{cases}\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0 \quad (r<a,\ 0<\theta<\alpha) \\ u(r,0)=u(r,\alpha)=0 \\ u(a,\theta)=f(\theta)\end{cases}$

设 $u(r,\theta)=R(r)\Theta(\theta)$

$$-\frac{r^2R''+rR'}{R}=\frac{\Theta''}{\Theta}=-\lambda$$

$$r^2R''+rR'-\lambda R=0$$

得固有值问题 $\begin{cases}\Theta''+\lambda\Theta=0 \quad (0<\theta<\alpha) \\ \Theta(0)=\Theta(\alpha)=0\end{cases}$

∵边界条件是Ⅰ类，∴没有0固有值，只需考虑 $\lambda>0$ 情况

设 $\lambda=n^2>0$

$\Rightarrow \Theta(\theta)=A\cos n\theta+B\sin n\theta$ ~~$A=1,2,$~~

代入边界条件 $\Rightarrow \begin{cases}A=0 \\ \sin n\alpha=0 \Rightarrow n\alpha=k\pi, \quad k=1,2,\cdots\end{cases}$

$\Rightarrow \Theta(\theta)=B\sin\frac{k\pi\theta}{\alpha}$

∴固有值 $\lambda_k=\left(\frac{k\pi}{\alpha}\right)^2$，对应固有函数 $\Theta_k(\theta)=\sin\frac{k\pi\theta}{\alpha}$ $\quad k=1,2,\cdots$

解方程 $r^2R''+rR'-\lambda R=0$

设 $r=e^t$ $\Rightarrow \frac{d^2R}{dt^2}-n^2R=0$

$\Rightarrow R_n=C_ne^{nt}+D_ne^{-nt}=C_nr^n+D_nr^{-n}$

~~即 $R_k=C_kr^k+D_kr^{-k}$~~ 即 $R_k=C_nr^{\frac{k\pi}{\alpha}}+D_nr^{-\frac{k\pi}{\alpha}}$

$$\Rightarrow u(r,\theta)=\sum_{k=1}^{+\infty}\left(C_k r^{\frac{k\pi}{\alpha}}+D_k r^{-\frac{k\pi}{\alpha}}\right)\sin\frac{k\pi\theta}{\alpha}$$

$\because |R(0)|<\infty \quad \therefore \quad D_k=0$

$$\Rightarrow u(r,\theta)=\sum_{k=1}^{+\infty}C_k r^{\frac{k\pi}{\alpha}}\sin\frac{k\pi\theta}{\alpha}$$

$\because u(a,\theta)=f(\theta)$

$$\therefore \quad f(\theta)=\sum_{k=1}^{+\infty}C_k a^{\frac{k\pi}{\alpha}}\sin\frac{k\pi\theta}{\alpha}$$

$$=\sum_{k=1}^{+\infty}\frac{\langle f(\theta),\ \sin\frac{k\pi\theta}{\alpha}\rangle}{\langle \sin\frac{k\pi\theta}{\alpha},\ \sin\frac{k\pi\theta}{\alpha}\rangle}\cdot\sin\frac{k\pi\theta}{\alpha}$$

对照得: $C_k a^{\frac{k\pi}{\alpha}}=\dfrac{\langle f(\theta),\sin\frac{k\pi\theta}{\alpha}\rangle}{\langle \sin\frac{k\pi\theta}{\alpha},\sin\frac{k\pi\theta}{\alpha}\rangle}\cdot\cancel{\sin\frac{k\pi\theta}{\alpha}}$

$$=\frac{\int_0^{2\pi}f(\theta)\sin\frac{k\pi\theta}{\alpha}d\theta}{\int_0^{2\pi}\sin^2\frac{k\pi\theta}{\alpha}d\theta}=\frac{\int_0^{\alpha}f(\theta)\sin\frac{k\pi\theta}{\alpha}d\theta}{\int_0^{\alpha}\frac{1-\cos\frac{2k\pi\theta}{\alpha}}{2}d\theta}\cdot\cancel{\sin\frac{k\pi\theta}{\alpha}}$$

$$=\frac{2}{\alpha}\int_0^{\alpha}f(\theta)\sin\frac{k\pi\theta}{\alpha}d\theta\cdot\cancel{\sin\frac{k\pi\theta}{\alpha}}$$

$$\Rightarrow u(r,\theta)=\sum_{k=1}^{+\infty}C_k r^{\frac{k\pi}{\alpha}}\sin\frac{k\pi\theta}{\alpha}$$

$$=\sum_{k=1}^{+\infty}\left[\frac{2}{\alpha}\int_0^{\alpha}f(\theta)\sin\frac{k\pi\theta}{\alpha}d\theta\right]\left(\frac{r}{a}\right)^{\frac{k\pi}{\alpha}}\sin\frac{k\pi\theta}{\alpha}$$

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8. 一半径为a的半圆形平板，其圆周边界上的温度保持$u(a,\theta)=T\theta(\pi-\theta)$，而直径边界上的温度为0，板的上下侧面绝热，试求板内的稳定温度分布.

解：
$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0, & 0<\theta<\pi,\ 0<r<a \\ u(a,\theta)=T\theta(\pi-\theta) \\ u(r,0)=u(r,\pi)=0 \\ |u(0,\theta)|<+\infty \end{cases}$$

设$u(r,\theta)=R(r)\Theta(\theta)$

$$-\frac{r^2R''+rR'}{R}=\frac{\Theta''}{\Theta}=-\lambda \qquad r^2R''+rR'-\lambda R=0$$

得固有值问题：

$$\begin{cases} \Theta''+\lambda\Theta=0 & 0<\theta<\pi \\ \Theta(0)=\Theta(\pi)=0 \end{cases}$$

∵边界条件是I类，∴无0固有值，∴只需考虑$\lambda>0$

设$\lambda=n^2>0$

$$\Rightarrow \Theta(\theta)=A\cos n\theta+B\sin n\theta$$

代入边界条件 $\Rightarrow \begin{cases} A=0 \\ \sin n\pi=0 \end{cases}$　$n=1,2,\cdots$

取$B=1$　$\Rightarrow \Theta(\theta)=\sin n\theta$，$n=1,2,\cdots$ 对应固有值$\lambda=n^2$.

固定λ，解方程 $r^2R''+rR'-\lambda R=0$，设$r=e^t$

$$\Rightarrow \frac{d^2R}{dt^2}-n^2R=0$$

$$\Rightarrow R_n=C_ne^{nt}+D_ne^{-nt}=C_nr^n+D_nr^{-n}$$

$$\Rightarrow u(r,\theta)=\sum_{n=1}^{+\infty}(C_nr^n+D_nr^{-n})\sin n\theta$$

$\because |R(0)| < +\infty \quad \therefore D_n = 0$

$$\Rightarrow u(r,\theta) = \sum_{n=1}^{+\infty} C_n r^n \sin n\theta .$$

$\because u(a,\theta) = T\theta(\pi-\theta)$

$$\therefore T\theta(\pi-\theta) = \sum_{n=1}^{+\infty} C_n \cdot a^n \sin n\theta$$

$$= \sum_{n=1}^{+\infty} \frac{\langle T\theta(\pi-\theta), \sin n\theta \rangle}{\langle \sin n\theta, \sin n\theta \rangle} \cdot \sin n\theta$$

对照得, $C_n \cdot a^n = \dfrac{\langle T\theta(\pi-\theta), \sin n\theta \rangle}{\langle \sin n\theta, \sin n\theta \rangle}$

$$= \frac{\int_0^\pi T\theta(\pi-\theta)\sin n\theta \, d\theta}{\int_0^\pi \sin^2 n\theta \, d\theta} = \frac{\int_0^\pi T\theta(\pi-\theta)\sin n\theta \, d\theta}{\frac{\pi}{2}}$$

$$\Rightarrow C_n = \frac{1}{a^n} \cdot \frac{2}{\pi} \int_0^\pi T\theta(\pi-\theta)\sin n\theta \, d\theta$$

$$= \frac{4T}{\pi n^3 a^n}(1-\cos n\pi)$$

$$\therefore u(r,\theta) = \sum_{n=1}^{+\infty} \frac{4T}{\pi n^3 a^n}(1-\cos n\pi) \cdot r^n \cdot \sin n\theta .$$

10. 解非齐次定解问题.

(5) $\begin{cases} u_{tt}=u_{xx}+g & (g为常数) \\ u(t,0)=0,\ u_x(t,l)=E & (E为常数) \\ u(0,x)=Ex,\ u_t(0,x)=0 \end{cases}$

解: 首先找满足边界条件的特解,把边界条件化为齐次:

取 $\tilde{u}(t,x)=Ex$

设 $w=u-\tilde{u}$ 则 $\begin{cases} w_{tt}=w_{xx}+g \\ w(t,0)=0,\ w_x(t,l)=0 \\ w(0,x)=0,\ w_t(0,x)=0 \end{cases}$

取满足上述方程的特解 $\tilde{w}=-\frac{g}{2}x^2+\frac{g}{2}l^2x$ (该特解~~要求~~满足边界条件)

令 $v=w-\tilde{w}$ 得: $\begin{cases} v_{tt}=v_{xx} \\ v(t,0)=0,\ v_x(t,l)=0 \\ v(0,x)=-\frac{g}{2}x^2+glx,\ v_t(0,x)=0 \end{cases}$

设 $v(t,x)=T(t)X(x)$

代入泛定方程,则 $T''X=X''T$

$\therefore \frac{T''}{T}=\frac{X''}{X}$

设上式为 $-\lambda$. 则 得到固有值问题: $\begin{cases} X''+\lambda X=0, & 0<x<l \\ X(0)=0,\ X'(l)=0 \end{cases}$

及 $T''+\lambda T=0$

① $\lambda>0$ 时, $\lambda=n^2$, $X''+n^2X=0 \Rightarrow X=A\cos nx+B\sin nx$

$X'=-An\,\cancel{\cos}\sin nx+B\cdot n\cos nx$

代入边界条件: $\Rightarrow \begin{cases} A=0 \\ \cos nl=0 \end{cases}$ ~~B·t~~ ~~B·l=0~~

$\Rightarrow nl=\frac{\pi}{2}(2k+1)$, $k=0,1,\cdots$

$\Rightarrow n=\frac{\pi(2k+1)}{2l}$

$\Rightarrow \lambda_k=\left[\frac{\pi(2k+1)}{2l}\right]^2$, $k=0,1,\cdots$

~~$\Rightarrow \lambda_k=$~~ 对应固有函数 $X_k=\sin\left(\frac{\pi(2k+1)x}{2l}\right)$, $k=0,1,\cdots$

② $\lambda=0$ 时, $X''=0$ 设 $X=CX+D$. 则代入边界条件知 $\begin{cases} D=0 \\ C=0 \end{cases} \Rightarrow X=0$ (舍).

③ $\lambda<0$ 时, 令 $\lambda=-n^2<0$. 则 $X''=n^2X$

$\Rightarrow X=E\cdot e^{nx}+F\cdot e^{-nx}$

代入边界条件 $\begin{cases} X(0)=0 \Rightarrow E+F=0 \\ X'(l)=n\cdot Ee^{nl}-nF\cdot e^{-nl}=0 \end{cases} \Rightarrow E=F=0$ (舍).

综上, 固有值 $\lambda_k=\left[\frac{\pi(2k+1)}{2l}\right]^2$, $k=0,1,\cdots$

对应固有函数 $X_k=\sin\left[\frac{\pi(2k+1)x}{2l}\right]$, $k=0,1,\cdots$

固定 λ_k , 得: $T''+\lambda_k T=0$, $\lambda_k>0$

$\Rightarrow T_k(t)=M_k\cos\left[\frac{\pi(2k+1)}{2l}t\right]+N_k\sin\left[\frac{\pi(2k+1)}{2l}\cdot t\right]$, $k=0,1,\cdots$

$\Rightarrow v(t,x)=\sum_{k=0}^{\infty}\left[M_k\cos\left[\frac{\pi(2k+1)}{2l}\cdot t\right]+N_k\sin\left[\frac{\pi(2k+1)t}{2l}\right]\right]\cdot\sin\left[\frac{\pi(2k+1)x}{2l}\right]$

$\because v(0,x)=-\frac{g}{2}x^2+glx$; $v_t(0,x)=0$

$\therefore \begin{cases} -\frac{g}{2}x^2+glx=\sum_{k=0}^{\infty}M_k\cdot\sin\left[\frac{\pi(2k+1)\cdot x}{2l}\right] \\ N_k=0 \end{cases}$

$$\therefore M_k=\frac{\langle -\frac{g}{2}x^2+glx, X_n\rangle}{\langle X_n, X_n\rangle}=\frac{\int_0^l(-\frac{g}{2}x^2+glx)\cdot\sin\left(\frac{\pi(2k+1)x}{2l}\right)dx}{\int_0^l\sin^2\left(\frac{\pi(2k+1)x}{2l}\right)dx}$$

$$= \frac{g}{l}\int_0^l x(2l-x)\cdot\sin\left(\frac{\pi(2k+1)x}{2l}\right)\cdot dx$$

$$= \frac{g}{l}\cdot\left(\frac{-16l^3}{\pi^3(2k+1)^3}\right)$$

$$= -\frac{16gl^2}{(2k+1)^3\pi^3}$$

$$\therefore V(t,x)=\sum_{k=0}^{\infty}\left[\frac{-16gl^2}{(2k+1)^3\pi^3}\right]\cos\frac{\pi(2k+1)t}{2l}\sin\frac{\pi(2k+1)x}{2l}$$

$$\therefore u(t,x)=\hat{u}+\tilde{W}+V$$

$$= Ex+glx-\frac{g}{2}x^2-\frac{16gl^2}{\pi^3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}\cos\frac{\pi(2k+1)t}{2l}\sin\frac{\pi(2k+1)x}{2l}$$

11.12) $u(a,\theta)=u_1$, $\frac{\partial u(b,\theta)}{\partial n}=u_2$

求环域$a<r<b$内泊松方程$\Delta_2 u=A$（A为常数）的解.

解:
$$\begin{cases}\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=A\\ u(a,\theta)=u_1\\ \frac{\partial u(b,\theta)}{\partial n}=u_2\end{cases}$$

首先找满足边界条件的特解，把边界条件化为齐次：

取 $\tilde{u}=u_1-u_2a+u_2 r$

设 $\tilde{w}=u-\tilde{u}$，则
$$\begin{cases}w_{rr}+\frac{1}{r}w_r+\frac{1}{r^2}w_{\theta\theta}=A-(\tilde{u}_{rr}+\frac{1}{r}\tilde{u}_r+\frac{1}{r^2}\tilde{u}_{\theta\theta})=A-\frac{u_2}{r}\\ w(a,\theta)=0\\ \frac{\partial w(b,\theta)}{\partial n}=0\end{cases}$$

~~取满足上述方程的特解~~ $\cancel{\tilde{w}=\frac{1}{4}Ar^2-u_2 r}$

~~令~~ $\cancel{v=w-\tilde{w}}$ 得

~~得:~~ $\cancel{v_{rr}+\frac{1}{r}v_r+\frac{1}{r^2}v_{\theta\theta}}$

把w的方程拆分成

(I): $\begin{cases} w_{rr}+\frac{1}{r}w_r+\frac{1}{r^2}w_{\theta\theta}=0, & a<r<b \\ w(a,\theta)=0 \\ \frac{\partial w(b,\theta)}{\partial n}=0 \end{cases}$

(II): 对w的方程利用冲量原理:

$\tilde{w}_{rr}+\frac{1}{r}\tilde{w}_r+\frac{1}{r^2}\tilde{w}_{\theta\theta}=0, \quad a<r<b$

$\tilde{w}(a,\theta)=0$

$\frac{\partial \tilde{w}(b,\theta)}{\partial n}=$

取 $\tilde{u}=\frac{1}{4}Ar^2$ $\quad w=u-\tilde{u}$

则 $\begin{cases} w_{rr}+\frac{1}{r}w_r+\frac{1}{r^2}w_{\theta\theta}=0 \\ w(a,\theta)=u_1-\frac{1}{4}Aa^2, \quad \frac{\partial w(b,\theta)}{\partial n}=\cancel{u_2-\frac{1}{2}Ab}\ u_2-\frac{1}{2}Ab \end{cases}$

边值与θ无关. 则 $w=w(r)$ 则 $r^2R''+rR'=0$ (欧拉方程)

$\Rightarrow w(r,\theta)=A_0+B_0\ln r$

$\Rightarrow \begin{cases} A_0+B_0\ln a=u_1-\frac{1}{4}Aa^2 \\ \frac{B_0}{b}=u_2-\frac{1}{2}Ab \end{cases} \Rightarrow \begin{cases} B_0=-\frac{1}{2}Ab^2+b\cdot u_2 \\ A_0=-\frac{1}{4}Aa^2+\frac{1}{2}Ab^2\ln a+u_1-bu_2\ln a \end{cases}$

$\Rightarrow w(r,\theta)=u_1-\frac{1}{4}Aa^2+\frac{1}{2}Ab^2\ln a-\frac{1}{2}Ab^2\ln r+bu_2\ln r-bu_2\ln a$

$\Rightarrow u=w+\tilde{u}=u_1+\frac{1}{4}A(r^2-a^2)+(\frac{1}{2}Ab^2-bu_2)\ln a+(bu_2-\frac{1}{2}Ab^2)\ln r$

$=u_1+\frac{1}{4}A(r^2-a^2)+(bu_2-\frac{1}{2}Ab^2)\ln\frac{r}{a}$

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12. 设w_n是$J_0(2w_n)=0$的正实根，把函数$f(x)=\begin{cases}1 & 0<x<1\\ \frac{1}{2} & x=1\\ 0 & 1<x<2\end{cases}$展开成贝塞尔函数$J_0(w_n x)$的级数.

解：w_n及$J_0(2w_n)=0$对应如下零阶贝塞尔固有值问题的固有值和固有函数：

$$\begin{cases}x^2y''+xy'+\mu x^2y=0, & 0\le x\le 2\\ |y(0)|<+\infty, & y(2)=0\end{cases}$$

边界条件为I类边界条件.

查公式，$N_0^2=\int_0^2 xJ_0^2(w_n x)\,dx=\frac{a^2}{2}J_{v+1}^2(wa)$　$(a=2,\ v=0)$

$$=2J_1^2(2w_n)$$

级数展开后$f(x)=\sum_{n=1}^{+\infty}C_nJ_0(w_nx)$

$$C_n=\frac{\langle f(x),J_0(w_nx)\rangle}{\langle J_0(w_nx),J_0(w_nx)\rangle}=\frac{\int_0^2 xf(x)J_0(w_nx)\,dx}{N_0^2}$$

$$\int_0^2 xf(x)J_0(w_nx)\,dx$$

$$=\int_0^1 xJ_0(w_nx)\,dx \qquad \text{(换元: } s=w_nx)$$

$$=\int_0^{w_n}\frac{s}{w_n}J_0(s)\,d\left(\frac{s}{w_n}\right)$$

$$=\frac{1}{w_n^2}\int_0^{w_n}sJ_0(s)\,ds$$

$$=\frac{1}{w_n^2}\int_0^{w_n}(sJ_1(s))'\,ds=\frac{1}{w_n^2}sJ_1(s)\Big|_0^{w_n}=\frac{1}{w_n}J_1(w_n)$$

$$\therefore C_n=\frac{\frac{1}{w_n}\cdot J_1(w_n)}{2J_1^2(2w_n)}=\frac{J_1(w_n)}{2w_nJ_1^2(2w_n)}$$

$$\therefore f(x)=\sum_{n=1}^{+\infty}\frac{J_1(w_n)}{2w_nJ_1^2(2w_n)}\cdot J_0(w_nx).$$

16. 半径为R的无限长圆柱体的侧表面保持一定的温度u_0，柱内的初始温度为0，求柱内的温度分布。

解：由对称性，知温度分布与角度无关. ∴设 $u=u(t,r)$

写出定解问题：
$$\begin{cases} u_t=a^2\Delta u=a^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right) & (0\le r<R). \\ u(t,R)=u_0 \\ u(0,r)=0 \end{cases}$$

边界条件非齐次，无法直接分离变量，故先把边界条件齐次化.

∵有特解 $v(t,r)=u_0$ 代入 $w=u-v$

$$\therefore \begin{cases} w_t=a^2\left(w_{rr}+\frac{1}{r}w_r\right) & (0\le r\le R). \\ w(t,R)=0 \\ w(0,r)=-u_0 \end{cases}$$

设 $w(t,r)=T(t)\cdot\tilde{R}(r)$ 代入方程 得：$T'\tilde{R}=a^2\left(T\cdot\tilde{R}''+\frac{1}{r}T\cdot\tilde{R}'\right)$

$$\frac{T'}{a^2T}=\frac{r\tilde{R}''+\tilde{R}'}{r\tilde{R}}=-\lambda$$

$$\Rightarrow \quad T'+\lambda a^2T=0. \qquad \begin{cases} r\tilde{R}''+\tilde{R}'+\lambda r\tilde{R}=0 \\ |\tilde{R}(0)|<+\infty.\ \tilde{R}(R)=0 \end{cases}$$

由S-L定理及自然边界条件和第一类边界边界条件可知. 零不是固有值

得零阶固有值问题（贝塞尔）
$$\begin{cases} r^2\tilde{R}''+r\tilde{R}'+\lambda(r^2-0^2)\tilde{R}=0 \\ |\tilde{R}(0)|<+\infty,\ \tilde{R}(R)=0 \end{cases}$$

固有值 $\lambda_n=w_n^2>0$ ，$n=1,2,\cdots$ （w_n为$J_0(w_nR)=0$的第n个正实根）.

固有函数 $\underbrace{J_0(w_nr)}_{\tilde{R}_n(r)}$

$\because T' + \lambda a^2 T = 0 \qquad \lambda_n = W_n^2$

$\therefore T_n'(t) = -W_n^2 a^2 T(t)$

$\therefore T_n(t) = e^{-W_n^2 a^2 t}$

$\therefore w(t,r) = \sum_{n=1}^{+\infty} C_n e^{-W_n^2 a^2 t} J_0(W_n r).$

代入初始条件：~~u₀~~ $w(0,r) = -u_0$

$\therefore -u_0 = \sum_{n=1}^{+\infty} C_n J_0(W_n r)$

$$\therefore C_n = \frac{\langle -u_0, J_0(W_n r)\rangle}{\langle J_0(W_n r), J_0(W_n r)\rangle}$$

查公式可知：$\alpha \neq 0, \beta = 0$：$N_0^2 = \frac{a^2}{2} J_1^2(wa).$

$\therefore \langle J_0(W_n r), J_0(W_n r)\rangle = \frac{R^2}{2} J_1^2(W_n R)$

$\langle -u_0, J_0(W_n r)\rangle = \int_0^R -u_0 \cdot r J_0(W_n r)\,dr$ （换元：$s = W_n r$）

$$= -\int_0^{W_n R} u_0 \cdot \frac{s}{W_n} \cdot J_0(s)\, d\left(\frac{s}{W_n}\right)$$

$$= -\frac{u_0}{W_n^2} \int_0^{W_n R} s J_0(s)\, ds$$ [公式：$(x^\nu J_\nu)' = x^\nu J_{\nu-1}$]

$$= -\frac{u_0}{W_n^2} \int_0^{W_n R} [s J_1(s)]' ds$$

$$= -\frac{u_0}{W_n^2} s J_1(s) \Big|_0^{W_n R}$$

$$= -\frac{u_0}{W_n^2} \cdot W_n R J_1(W_n R) = -\frac{u_0 R}{W_n} J_1(W_n R)$$

$$\therefore C_n = \frac{-\frac{u_0 R}{W_n} J_1(W_n R)}{\frac{R^2}{2} J_1^2(W_n R)} = -\frac{2u_0}{R W_n} J_1^{-1}(W_n R).$$

$$\therefore w(t,r) = \sum_{n=1}^{+\infty} \left(-\frac{2u_0}{R W_n} J_1^{-1}(W_n R)\right) \cdot e^{-W_n^2 a^2 t} J_0(W_n r).$$

$$\therefore u(t,r) = u_0 - \frac{2u_0}{R} \sum_{n=1}^{+\infty} \frac{1}{W_n \cdot J_1(W_n R)} e^{-W_n^2 a^2 t} J_0(W_n r).$$

（W_n为$J_0(W_n R)$的第n个正实根）

18(1). 解定解问题：

$$\begin{cases} u_{rr}+\frac{1}{r}u_r+u_{zz}=0 & (0<r<a,\ 0<z<l). \\ u(a,z)=0 \\ u(r,0)=0,\quad u(r,l)=T_0\ (常数); \end{cases}$$

解：分离变量 $u(r,z)=R(r)Z(z)$.

$$\therefore\ R''Z+\frac{1}{r}R'Z+RZ''=0$$

$$\Rightarrow\ -\frac{Z''}{Z}=\frac{rR''+R'}{rR}$$ 令其为 $(-\mu)$.

$$\therefore\ \begin{cases} r^2R''+rR'+\mu r^2R=0. & 0<r<a \\ |R(0)|<+\infty,\quad R(a)=0 \end{cases}$$

为零阶贝塞尔固有值问题，因为有一个条件是I类边界条件，因而零不是固有值.

设 $\underbrace{\mu=w_n^2}_{0<w_1<w_2<\cdots}$. 对应固有函数 $J_0(w_nr)$. 代入Z的方程

$$\Rightarrow\ Z''-w_n^2Z=0$$

$$\Rightarrow\ Z_n=A_ne^{w_nz}+B_ne^{-w_nz}$$

$$\therefore\ u(r,z)=\sum_{n=1}^{\infty}(A_ne^{w_nz}+B_ne^{-w_nz})J_0(w_nr)$$

代入初始条件：

$$\begin{cases} A_n+B_n=0 \\ \sum_{n=1}^{\infty}(A_ne^{w_nl}+B_ne^{-w_nl})J_0(w_nr)=T_0 \end{cases}$$

$$\therefore\ A_ne^{w_nl}+B_ne^{-w_nl}=\frac{\langle T_0,J_0(w_nr)\rangle}{\langle J_0(w_nr),J_0(w_nr)\rangle}$$

查公式. $\alpha\neq0,\ \beta=0$. $N_0^2=\frac{a^2}{2}J_1^2(wa)$

$$\therefore\ \langle J_0(w_nr),J_0(w_nr)\rangle=\frac{a^2}{2}J_1^2(w_na).$$

$$\langle T_0,J_0(w_nr)\rangle=\int_0^a T_0J_0(w_nr)\cdot r\,dr=\frac{T_0a}{w_n}J_1(w_na)$$

$$\therefore A_n e^{w_n l} + B_n e^{-w_n l} = \frac{\frac{T_0 a}{w_n} J_1(w_n a)}{\frac{a^2}{2} J_1^2(w_n a)} = \frac{2T_0}{a w_n J_1(w_n a)}$$

$$\because A_n + B_n = 0$$

$$\therefore A_n = \frac{2T_0}{(e^{w_n l} - e^{-w_n l})\, a w_n J_1(w_n a)}$$

$$\therefore u(r,z) = \sum_{n=1}^{\infty} \cdot \frac{2T_0 \quad (e^{w_n z} - e^{-w_n z})}{(e^{w_n l} - e^{-w_n l})\, a w_n J_1(w_n a)} \cdot J_0(w_n r)$$

(w_n 为 $J_0(w_n R)=0$ 的第n个正实根).

第三章　24(1)(3)　.26.　28.　　　　第四章　1.(1)　(2)

先观察

24. 把下列函数按勒让德函数系展开.

(1) 为奇函数. 展开只有奇次项

(1) $f(x)=x^3$　　(3) $f(x)=|x|$.

(2) --偶--　　--偶--

解:(1) 勒让德多项式 $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

$\therefore P_0=1 \quad P_1=x \quad P_2=\frac{1}{2}(3x^2-1) \quad P_3=\frac{1}{2}(5x^3-3x)$

$\therefore x^3=C_0 P_1(x)+C_3P_3(x)$

~~$x^3=C_0+\frac{C_3}{2}(5x^3-3x)$~~

$x^3=C_0\cdot x+\frac{C_3}{2}(5x^3-3x)$

$\therefore 2x^3=2C_0x+5C_3x^3-3C_3x$

$\therefore C_0=\frac{3}{5} \qquad C_3=\frac{2}{5}$

$\therefore f(x)=\frac{3}{5}P_1(x)+\frac{2}{5}P_3(x)$

(3) $f(x)=\sum_{n=0}^{\infty}\frac{\langle f,P_n\rangle}{\langle P_n,P_n\rangle}P_n$

$\langle P_n,P_n\rangle=\frac{2}{2n+1}$

$\because f(x)=|x|$ 是偶函数, 则 $f(x)$ 展开成 $\{P_{2n}\}$ 的线性组合

$\therefore f(x)=\sum_{n=0}^{\infty}\frac{\langle f,P_{2n}\rangle}{\langle P_{2n},P_{2n}\rangle}P_{2n} \qquad \langle P_{2n},P_{2n}\rangle=\frac{2}{4n+1}$

$\therefore \langle f,P_{2n}\rangle=\int_{-1}^{1}|x|\cdot P_{2n}(x)dx$ ~~$=2\int_0^1 x\cdot P_{2n}(x)dx=2\int_0^1 x\cdot\frac{P_{2n+1}'-P_{2n-1}'}{4n+1}dx$~~

$\int_0^1 x^m P_n(x)dx=\frac{m}{m+n+1}\int_0^1 x^{m-1}P_{n-1}(x)dx$

$=2\int_0^1 x\cdot P_{2n}(x)dx=\frac{2}{2n+2}\int_0^1 P_{2n-1}(x)dx$

$=\frac{1}{n+1}\cdot\frac{1}{2(2n-1)+1}\int_0^1[P_{2n-1+1}'(x)-P_{2n-1-1}'(x)]dx$

$=\frac{P_{2n-2}(0)-P_{2n}(0)}{(n+1)(4n-1)} \quad (2n-2\geq 0\text{ 即 }n\geq 1).$

当 $n=0$ 时, $\langle f,P_0(x)\rangle=\int_{-1}^{1}|x|\cdot 1\,dx=1 \qquad \langle P_0,P_0\rangle=2.$

$\therefore |x|=\sum_{n=0}^{\infty}\frac{\langle f,P_{2n}\rangle}{\langle P_{2n},P_{2n}\rangle}P_{2n}=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(4n+1)\cdot[P_{2n-2}(0)-P_{2n}(0)]}{2(n+1)(4n-1)}\cdot P_{2n}(x)$

26. 在半径为1的球内求调和函数u，使$u|_{r=1}=3\cos 2\theta+1$.

解：取极坐标(r,θ,φ) . $r\in[0,1]$. $\theta\in[0,\pi]$, $\varphi\in[0,2\pi)$.

表面温度与φ无关，由对称性，球内温度分布也与φ无关.

因而设$u=u(r,\theta)$. 定解问题：

$$\begin{cases}\Delta u=0, & 0<r<1, \theta\in[0,\pi]\\ u|_{r=1}=3\cos 2\theta+1\end{cases}$$

易知 $u(r,\theta)=\sum_{n=0}^{\infty}(A_n r^n+B_n r^{-n-1})P_n(\cos\theta)$

$\because$ 球心处温度有限 $\therefore B_n=0$

$\therefore u(r,\theta)=\sum_{n=0}^{\infty}A_n r^n P_n(\cos\theta)$

$\because u(1,\theta)=3\cos 2\theta+1=\sum_{n=0}^{\infty}A_n P_n(\cos\theta)$

$\therefore 6\cos^2\theta-2=\sum_{n=0}^{\infty}A_n P_n(\cos\theta)$.

$\therefore 6x^2-2=\sum_{n=0}^{\infty}A_n P_n(x)$.

$6x^2-2=A_0+A_2P_2(x)$

$6x^2-2=A_0+A_2\cdot\frac{1}{2}(3x^2-1)$

$\therefore A_0=0$, $A_2=4$ $\therefore u(r,\theta)=4r^2P_2(\cos\theta)=4r^2\cdot\frac{1}{2}(3\cos^2\theta-1)$

$=2r^2(3\cos^2\theta-1)$ ~~$=6r^2(\cos^2\theta-1)$~~

28. 半球的球面保持一定的温度u_0，试分别在下列条件下，求这个半球内的稳定温度分布：

(1) 半球底面保持常温零度

(2) 半球底面绝热

解：(1) 把半球补成一个完整的球，要使底部恒温0，仅需球表面温度上、下反对称

即 $\begin{cases} \Delta u = 0, & 0 \le r \le a, \ \theta \in [0,\pi] \\ u|_{r=a} = u_0 & \theta \in [0,\frac{\pi}{2}] \\ u|_{r=a} = -u_0 & \theta \in [\frac{\pi}{2},\pi] \end{cases}$ 设半径为 a.

易知：$u = \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-n-1}) P_n(\cos\theta)$

$\because |u(0,\theta)| < \infty \quad \therefore B_n = 0$

$\therefore u = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$

$\therefore \sum_{n=0}^{\infty} A_n a^n P_n(\cos\theta) = \begin{cases} u_0, & \theta \in [0,\frac{\pi}{2}] \\ -u_0, & \theta \in [\frac{\pi}{2},\pi] \end{cases}$

即 $\sum_{n=0}^{\infty} A_n a^n P_n(x) = f(x) = \begin{cases} u_0, & \overset{x}{\cancel{\cos\theta}} \in [0,1] \\ -u_0, & \overset{x}{\cancel{\cos\theta}} \in [-1,0] \end{cases}$

将 $f(x)$ 按勒让德多项式分解：

$f(x) = \sum_{n=0}^{\infty} C_n P_n(x) \qquad C_n = A_n \cdot a^n$

$C_n = \dfrac{\langle f(x), P_n(x) \rangle}{\langle P_n(x), P_n(x) \rangle} \qquad \langle P_n(x), P_n(x) \rangle = \dfrac{2}{2n+1}$

$\because f(x)$ 为奇函数，$\therefore$ 当 n 为偶数时 $\langle f(x), P_n(x) \rangle = \int_{-1}^{1} f(x) P_n(x)\,dx = 0$.

$\therefore n$ 为奇数 $\quad n = 2k+1 \quad k \ge 0$.

$\therefore \langle f(x), P_{2k+1}(x) \rangle = 2\int_0^1 u_0 \cdot P_{2k+1}(x)\,dx = 2u_0 \cdot \int_0^1 \dfrac{P_{2k+2}'(x) - P_{2k}'(x)}{2(2k+1)+1}\,dx$

$= \dfrac{2u_0 (P_{2k}(0) - P_{2k+2}(0))}{4k+3}$

又 $\because \langle P_{2k+1}(x), P_{2k+1}(x) \rangle = \dfrac{2}{4k+3}$

$\therefore u_0 [P_{2k}(0) - P_{2k+2}(0)] = A_{2k+1} \cdot a^{2k+1}$

$\therefore u = \sum_{k=0}^{\infty} \left(\dfrac{r}{a}\right)^{2k+1} \cdot u_0 [P_{2k}(0) - P_{2k+2}(0)] \cdot P_{2k+1}(\cos\theta)$

1. 用Fourier变换解下列定解问题：

(1) $\begin{cases} \Delta_2 u=0 \quad (-\infty<x<+\infty,\ y>0) \\ u(x,0)=f(x) \\ \text{当 } x^2+y^2\to+\infty \text{ 时},\ u(x,y)\to 0 \end{cases}$

(2) $\begin{cases} u_t=a^2u_{xx}+f(t,x) \quad (t>0,\ -\infty<x<+\infty) \\ u(0,x)=0 \end{cases}$

解：(1) 作Fourier变换 $\bar{u}(\lambda,y)=\int_{-\infty}^{+\infty}u(x,y)e^{i\lambda x}dx$

∵ ∴ $\bar{u}_{xx}=(-i\lambda)^2\bar{u}=-\lambda^2\bar{u}$

∴ $\begin{cases} -\lambda^2\bar{u}+\bar{u}_{yy}=0, \quad -\infty<\lambda<+\infty,\ y>0 \\ \bar{u}(\lambda,0)=F[f(x)] \end{cases}$

∵ $\bar{u}_{yy}=\lambda^2\bar{u}$

∴ $\bar{u}(\lambda,y)=C_1(\lambda)e^{-\lambda y}+C_2(\lambda)e^{\lambda y}$

~~又∵ $\bar{u}(\lambda,0)=F[f(x)]$ ∴ $C_1(\lambda)+C_2(\lambda)=F[f(x)]$~~

∵ $\lambda^2+y^2\to+\infty$ 时，$\bar{u}(\lambda,y)\to 0$ ∴ $C_2(\lambda)=0$. X

∴ $\bar{u}(\lambda,y)=C_1(\lambda)e^{-\lambda y}$

又∵ $\bar{u}(\lambda,0)=F[f(x)]$ ∴ $C_1(\lambda)=F[f(x)]$

∴ $\bar{u}(\lambda,y)=F[f(x)]\cdot e^{-\lambda y}$

∴ $u(x,y)=F^{-1}[F[f(x)]]*F^{-1}[e^{-\lambda y}]=f(x)*F^{-1}[e^{-\lambda y}]$

∵ $F^{-1}[e^{-\lambda y}]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-\lambda y}\cdot e^{-i\lambda x}d\lambda$

$\because \lambda^2+y^2 \to +\infty$ 时，$\bar{u}(\lambda,y) \to 0$

$\therefore \lambda>0$ 时，$C_2(\lambda)=0$ ，$\lambda<0$ 时，$C_1(\lambda)=0$

$$\therefore \bar{u}(\lambda,y)=\begin{cases} C_1(\lambda)e^{-\lambda y}, & \lambda>0, y>0 \\ C_2(\lambda)e^{\lambda y}, & \lambda<0, y>0. \end{cases}$$

$\because \bar{u}(\lambda,0)=\cancel{f(x)}\ F[f(x)]$

$$\cancel{\therefore C_1(\lambda)=\begin{cases} F[f(x)], & \lambda>0,y>0 \\ 0, & \lambda<0,y>0 \end{cases}} \quad \cancel{C_2(\lambda)=\begin{cases} 0, & \lambda>0,y>0 \\ F[f(x)], & \lambda<0,y>0 \end{cases}}$$

$$\therefore \bar{u}(\lambda,y)=\begin{cases} F[f]\cdot e^{-\lambda y}, & \lambda>0, y>0 \\ F[f]\cdot e^{\lambda y}, & \lambda<0, y>0 \end{cases}$$

$\cancel{\therefore u(x,y)=\frac{1}{2\pi}\left(\int_0^\infty F[f]\right.}$

$$\therefore u(x,y)=f(x)*F^{-1}[e^{-\lambda y}\cdot h(\lambda)]+f(x)*F^{-1}[e^{\lambda y}(1-h(\lambda))]$$

其中，定义：$h(\lambda)=\begin{cases} 1, & \lambda>0 \\ 0, & \lambda<0 \end{cases}$

$$\therefore F^{-1}[e^{-\lambda y}h(\lambda)]=\frac{1}{2\pi}\int_0^\infty e^{-\lambda y}\cdot e^{-i\lambda x}d\lambda=\frac{1}{2\pi}\left(-\frac{1}{y+ix}\cdot e^{-(y+ix)\cdot\lambda}\right)\Big|_0^{+\infty}$$

$$=\frac{1}{2\pi}\cdot\frac{1}{y+ix}$$

$$F^{-1}[e^{\lambda y}(1-h(\lambda))]=\frac{1}{2\pi}\int_{-\infty}^0 e^{\lambda y}\cdot e^{-i\lambda x}d\lambda=\frac{1}{2\pi}\left(\frac{1}{y-ix}\cdot e^{(y-ix)\lambda}\right)\Big|_{-\infty}^0$$

$$=\frac{1}{2\pi}\cdot\frac{1}{y-ix}$$

$$\therefore u(x,y)=\frac{1}{2\pi}\left(f*\frac{1}{y+ix}+f*\cdot\frac{1}{y-ix}\right)=\frac{y}{\pi}\left(f*\frac{1}{x^2+y^2}\right)$$

$$\therefore u(x,y)=\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{f(s)}{(x-s)^2+y^2}ds.$$

$\bar{u}(t,\lambda)$

(2) 作Fourier 变换 $\bar{u} = \int_{-\infty}^{+\infty} u(t,x) e^{i\lambda x} dx$

$\therefore \bar{u}_{xx} = (-i\lambda)^2 \bar{u} = -\lambda^2 \bar{u}$

$\therefore \begin{cases} \bar{u}_t = -\lambda^2 a^2 \bar{u} + \bar{f}(t,\lambda) \\ \bar{u}(0,\lambda) = 0 \end{cases}$

~~$\therefore \bar{u}(t,\lambda) = C(\lambda) e^{-\lambda^2 a^2 t}$~~

$\because \bar{u}_t + \lambda^2 a^2 \bar{u} = \bar{f}(t,\lambda)$

利用一阶常微分方程通解：

$y' + p(x) y = Q(x) \Rightarrow y(x) = e^{-\int p dx} \left(\int e^{\int p dx} Q dx + C \right)$

$\therefore \bar{u} = e^{-\lambda^2 a^2 t} \left(\int_0^t e^{\lambda^2 a^2 \tau} \cdot \bar{f}(\tau,\lambda) d\tau + C(\lambda) \right)$

$\because \bar{u}(0,\lambda) = 0$

$\therefore C(\lambda) = 0$

$\therefore \bar{u}(t,\lambda) = \int_0^t \bar{f}(\tau,\lambda) \cdot e^{\lambda^2 a^2 (\tau - t)} d\tau$

$\therefore u(t,x) = F^{-1}[\bar{u}(t,\lambda)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\int_0^t F[f] \cdot e^{\lambda^2 a^2 (\tau - t)} d\tau \right] e^{-i\lambda x} d\lambda$

$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\int_0^t \bar{f}(\tau,\lambda) \cdot e^{\lambda^2 a^2 (\tau - t)} d\tau \right] e^{-i\lambda x} d\lambda$

$= \int_0^t \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \bar{f}(\tau,\lambda) \cdot e^{\lambda^2 a^2 (\tau - t)} \cdot e^{-i\lambda x} d\lambda \right) d\tau$

$= \int_0^t F^{-1}\left[\bar{f}(\tau,\lambda) \cdot e^{\lambda^2 a^2 (\tau - t)} \right] d\tau$

$= \int_0^t F^{-1}[\bar{f}(\tau,\lambda)] * F^{-1}\left[e^{\lambda^2 a^2 (\tau - t)} \right] d\tau$

$= \int_0^t f(\tau,\lambda) * F^{-1}\left[e^{\lambda^2 a^2 (\tau - t)} \right] d\tau$

$= \int_0^t \left[f(\tau,\lambda) * \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{\lambda^2 a^2 (\tau - t)} e^{-i\lambda x} d\lambda \right) \right] d\tau$

$$\because \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\lambda^2 a^2(\tau-t)} e^{-i\lambda x} d\lambda \qquad \text{令 } a' = a^2(t-\tau)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\lambda^2 a'} \cdot e^{-i\lambda x} d\lambda$$

$$= \frac{1}{2\sqrt{\pi a'}} \cdot e^{-\frac{\lambda^2}{4a'}}$$

$$= \frac{1}{2\sqrt{\pi a^2(t-\tau)}} \cdot e^{-\frac{\lambda^2}{4a^2(t-\tau)}}$$

$$= \frac{1}{2a\sqrt{\pi(t-\tau)}} e^{\frac{-\lambda^2}{4a^2(t-\tau)}}$$

$$\therefore \quad u(t,x) = \int_0^t \left[f(\tau,\lambda) * \left(\frac{1}{2a\sqrt{\pi(t-\tau)}} e^{-\frac{\lambda^2}{4a^2(t-\tau)}} \right) \right] d\tau$$

$$= \int_0^t \left[\int_{-\infty}^{+\infty} f(\tau,\xi) \frac{1}{2a\sqrt{\pi(t-\tau)}} e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}} d\xi \right] d\tau$$

$$= \frac{1}{2a\sqrt{\pi}} \int_0^t \int_{-\infty}^{+\infty} \frac{f(\tau,\xi)}{\sqrt{t-\tau}} \exp\left\{ -\frac{(x-\xi)^2}{4a^2(t-\tau)} \right\} d\xi \, d\tau$$

第五章　1(1)　3(2)　4(1)

1. 证明下列公式：$x\delta(x)=0$.

证：$\because \delta(x)=\begin{cases}0, & x\neq 0\\ +\infty, & x=0\end{cases}$

$\therefore x\delta(x)=\cancel{0\cdot\delta(x)}\ 0$

3.(2) 解定解问题：

$$\begin{cases}u_{tt}=a^2u_{xx} & (0<x<l,\ t>0)\\ u_x(t,0)=u_x(t,l)=0.\\ u(0,x)=0,\ u_t(0,x)=\delta(x-\xi) & (0<\xi<l)\end{cases}$$

解：设 $u(t,x)=T(t)X(x)$

则 $T''X=a^2TX''$

设 $\frac{T''}{a^2T}=\frac{X''}{X}=-\lambda$.

得到固有值问题：$\begin{cases}X''+\lambda X=0 & (0<x<l)\\ X'(0)=X'(l)=0\end{cases}$

和. $T''+\lambda a^2T=0$

① $\lambda<0$，舍去

② $\lambda=0$ 时，$X''=0$　$X=A_0x+B_0$

~~由初始条件~~ $A_0=0$　$\cancel{\therefore X}$

$\therefore \lambda_0=0$ 对应 固有函数 $X_0=1$

③ $\lambda>0$ 时，$X''+\lambda X=0$

$X=A_n\cos\sqrt{\lambda}\cdot x+B_n\sin\sqrt{\lambda}\cdot x$

$X'=-A_n\sqrt{\lambda}\cdot\sin\sqrt{\lambda}\cdot x+B_n\sqrt{\lambda}\cos\sqrt{\lambda}x$

$\because$ ~~由初始条件~~ $\because X'(0)=X'(l)=0$

$\therefore \quad B_n=0 \quad \sqrt{\lambda}\cdot l=n\pi. \quad \lambda_n=\left(\frac{n\pi}{l}\right)^2, \ n=1,2,\cdots$

$\therefore \quad \lambda_n=\left(\frac{n\pi}{l}\right)^2$ 对应固有函数 $X_n=\cos\frac{n\pi x}{l}. \quad n=1,2,\cdots$

固定 λ_n. $\quad T_n''+\lambda_n a^2 T_n=0$

$\therefore T_n=C_n\cos\sqrt{\lambda_n a^2}\,t+D_n\sin\sqrt{\lambda_n a^2}\,t$

$=C_n\cos\frac{n\pi a t}{l}+D_n\sin\frac{n\pi a t}{l}, \quad n=1,2,\cdots$

$T_0''=0 \quad T_0=C_0+D_0\cdot t$

$\therefore u(t,x)=C_0+D_0 t+\sum_{n=1}^{+\infty}\left(C_n\cos\frac{n\pi a t}{l}+D_n\sin\frac{n\pi a t}{l}\right)\cdot\cos\frac{n\pi x}{l}, \quad n=1,2,\cdots$

由初始条件: $u(0,x)=0 \quad u_t(0,x)=\delta(x-\xi)$

$u(0,x)=0 \Rightarrow C_0=0, \ C_n=0$

$\therefore u(t,x)=D_0\cdot t+\sum_{n=1}^{\infty}D_n\cdot\sin\frac{n\pi a t}{l}\cdot\cos\frac{n\pi x}{l}, \quad n=1,2,\cdots$

$\therefore u_t(t,x)=D_0+\sum_{n=1}^{\infty}D_n\cdot\frac{n\pi a}{l}\cos\frac{n\pi a t}{l}\cdot\cos\frac{n\pi x}{l}$

$\therefore u_t(0,x)=D_0+\sum_{n=1}^{\infty}D_n\cdot\frac{n\pi a}{l}\cos\frac{n\pi x}{l}=\delta(x-\xi)$

$$\therefore D_n\cdot\frac{n\pi a}{l}=\frac{\langle\delta(x-\xi),X_n\rangle}{\langle X_n,X_n\rangle}=\frac{\int_0^l\delta(x-\xi)\cdot\cos\frac{n\pi x}{l}\,dx}{\int_0^l\cos^2\frac{n\pi x}{l}\,dx}=\frac{\cos\frac{n\pi\xi}{l}}{\frac{l}{2}}=\frac{2}{l}\cos\frac{n\pi\xi}{l}$$

$(n=1,2,\cdots)$

$$D_0=\frac{\langle\delta(x-\xi),X_0\rangle}{\langle X_0,X_0\rangle}=\frac{\int_0^l\delta(x-\xi)\cdot dx}{\int_0^l 1^2\,dx}=\frac{1}{l}.$$

$$\therefore u(t,x)=\frac{1}{l}\cdot t+\sum_{n=1}^{\infty}\frac{2}{n\pi a}\cos\frac{n\pi\xi}{l}\sin\frac{n\pi a t}{l}\cos\frac{n\pi x}{l}$$

4.(1) 利用拉普拉斯方程的基本解．求下列方程的基本解：

$$u_{xx}+\beta^2 u_{yy}=0 \quad (\beta>0,\ \beta\text{为常数})$$

解：$u_{xx}+\beta^2 u_{yy}=\delta(x,y)$

设 $u(x,y)=w(x,\frac{1}{\beta}y)$

则 $w_{xx}+w_{yy}=\delta(x,\frac{1}{\beta}y)=\frac{1}{\beta}\delta(x,y)$

∵ 二维拉普拉斯方程基本解为 $\frac{1}{2\pi}\ln r$

$$\therefore\ w(x,y)=\frac{1}{\beta}\cdot\frac{1}{2\pi}\ln r=\frac{1}{\beta}\cdot\frac{1}{2\pi}\ln\sqrt{x^2+y^2}$$

$$\therefore\ w(x,\tfrac{1}{\beta}y)=\frac{1}{2\pi\beta}\ln\sqrt{x^2+\frac{1}{\beta^2}y^2}$$

$$\because\ u(x,y)=w(x,\tfrac{1}{\beta}y)$$

$$\therefore\ u(x,y)=\frac{1}{2\pi\beta}\ln\sqrt{x^2+\frac{1}{\beta^2}y^2}$$

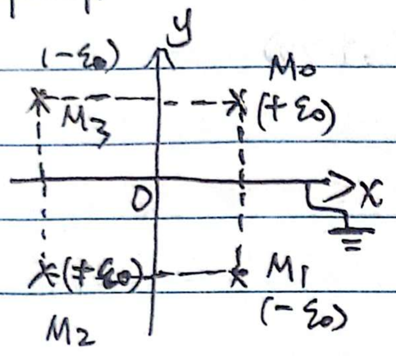
第五章　7. 8. 9(1)

7. 求下列平面区域内第一边值问题的格林函数：

(1) 四分之一平面：$x>0$，$y>0$。

(2) 二分之一单位圆内：$x^2+y^2<1$，$y>0$。

解：(1)



设 M_0 为 $x>0, y>0$ 内的点，在 M_0 放置 $+\varepsilon_0$ 点电荷

由对称性作出 M_1、M_2、M_3 三个点，并~~在此处~~分别放置 $-\varepsilon_0$、$+\varepsilon_0$、$-\varepsilon_0$ 三~~点~~个点电荷。

令 $M(x,y)$、$M_0(\xi,\eta)$

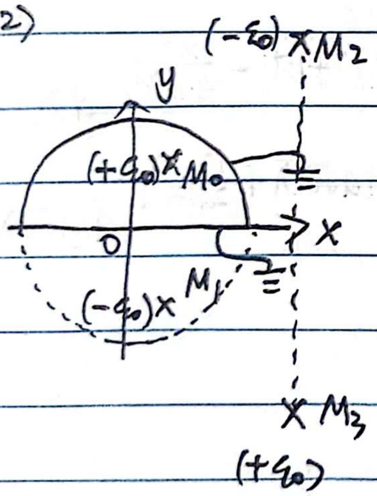
则 $M_1(\xi,-\eta)$，$M_2(-\xi,-\eta)$，$M_3(-\xi,\eta)$

令 $r_i=\overline{MM_i}$ $(i=0,1,2,3)$

$$\therefore G(M;M_0)=-\frac{1}{2\pi}(\ln r_0+\ln r_2-\ln r_1-\ln r_3)$$

$$=\frac{1}{2\pi}\ln\frac{r_1r_3}{r_0r_2}.$$

(2)



设 M_0 为 $x^2+y^2<1, y>0$ 内的点。在 M_0 放置 $+\varepsilon_0$ 点电荷

由对称性作出 M_1、M_2、M_3 三个点，并分别放置 $-\varepsilon_0$、$-\varepsilon_0$、$+\varepsilon_0$

三个点电荷。

令 $M(x,y)$　$M_0(\xi,\eta)$，$\rho_0=\sqrt{\xi^2+\eta^2}$

则 ~~M_0~~ $M_1(\xi,-\eta)$ ~~$M_2($~~

$\because \begin{cases}\rho_2\cdot\rho_0=R^2=1\\ O, M_0, M_2 \text{共线}\end{cases}$ $\therefore M_2\left(\frac{\xi}{\rho_0^2},\frac{\eta}{\rho_0^2}\right)$　$M_3\left(\frac{\xi}{\rho_0^2},-\frac{\eta}{\rho_0^2}\right)$

$$\therefore G'(M;M_0)=-\frac{1}{2\pi}(\ln|MM_0|-\ln|MM_1|+\ln|MM_3|-\ln|MM_2|)$$

$$=\frac{1}{2\pi}\ln\frac{|MM_1|\cdot|MM_2|}{|MM_0|\cdot|MM_3|}$$

又∵当M在圆周上时，圆周上的电势不为0

$$\therefore G(M;M_0)=\frac{1}{2\pi}\ln\frac{|MM_1|\cdot|MM_2|}{|MM_0|\cdot|MM_3|}-\frac{1}{2\pi}\ln\frac{R}{\rho_0}+\frac{1}{2\pi}\ln\frac{R}{\rho_0}=\frac{1}{2\pi}\ln\frac{|MM_1||MM_2|}{|MM_0||MM_3|}$$

18. 求方程 $u_t = a^2 u_{xx} + bu$ 的柯西问题的基本解

解：即求定解问题 $\begin{cases} u_t = a^2 u_{xx} + bu, & (-\infty < x < +\infty;\ t > 0) \\ u(0,x) = \delta(x) \end{cases}$

作 Fourier 变换：$\bar{u}(t,\lambda) = \int_{-\infty}^{+\infty} u(t,x) e^{i\lambda x} dx$

则有 $\begin{cases} \bar{u}_t = -a^2\lambda^2 \bar{u} + b\bar{u}, & (-\infty < \lambda < +\infty,\ t > 0) \\ \bar{u}(0,\lambda) = 1 \end{cases}$

$\therefore \quad \bar{u} = C(\lambda) e^{(-a^2\lambda^2 + b)t}$

令 $t=0$，则 $C(\lambda) = 1$

$\therefore \bar{u} = e^{(-a^2\lambda^2 + b)t}$

作 Fourier 反变换：

$$u(t,x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{(-a^2\lambda^2+b)t} \cdot e^{-i\lambda x} d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-(a^2\lambda^2 t + i\lambda x) + bt} d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-(a\sqrt{t}\cdot\lambda + \frac{ix}{2a\sqrt{t}})^2 + bt - \frac{x^2}{4a^2 t}} d\lambda$$

$$= \frac{1}{2\pi} e^{bt - \frac{x^2}{4a^2 t}} \int_{-\infty}^{+\infty} e^{-(a\sqrt{t}\lambda + \frac{ix}{2a\sqrt{t}})^2} d(a\sqrt{t}\lambda) \cdot \frac{1}{a\sqrt{t}}$$

$$= \frac{1}{2\pi} e^{bt - \frac{x^2}{4a^2 t}} \cdot \frac{\sqrt{\pi}}{a\sqrt{t}}$$

$$= \frac{e^{bt - \frac{x^2}{4a^2 t}}}{2a\sqrt{\pi t}}$$

9. 用基本解方法求下列柯西问题：

$$\begin{cases} u_t + a u_x = f(t,x) & (t>0,\ -\infty<x<+\infty) \\ u|_{t=0} = \varphi(x) \end{cases}$$

解：该题基本解对应定解问题：

$$\begin{cases} W_t + a W_x = 0 & (t>0,\ x\in R) \\ w|_{t=0} = \delta(x). \end{cases}$$

作~~作~~Fourier变换：$\bar{w}(t,\lambda) = \int_{-\infty}^{+\infty} w(t,x) e^{i\lambda x} dx$

$\therefore \begin{cases} \bar{w}_t - i\lambda a \bar{w} = 0 \\ \bar{w}|_{t=0} = 1 \end{cases}$

$\therefore \quad \bar{w}(t,\lambda) = C(\lambda) e^{i\lambda a t}$

$\because t=0,\ \bar{w}=1 \quad \therefore C(\lambda)=1$

$\therefore \bar{w}(t,\lambda) = e^{i\lambda a t}$

作 Fourier反变换 $w(t,x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\lambda a t}\cdot e^{-i\lambda x} d\lambda$

$\cancel{= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\lambda(at+x)} dx}$

$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\lambda(x-at)} d\lambda.$

$= \delta(x-at)$

代入公式： 其中 $U(t,x) = \delta(x-at)$ 为基本解 $M=x.$

$\therefore u(t,x) = U(t,M) * \varphi(M) + \int_0^t U(t-\tau,M) * f(\tau,M)\, d\tau$

$= U(t,x) * \varphi(x) + \int_0^t U(t-\tau,x) * f(\tau,x)\, d\tau$

$\cancel{= \int_{-\infty}^{+\infty} \delta(\xi-at)\varphi(x-\xi)d\xi + \int_0^t \delta(x-at) * f(\tau,x) d\tau}$

$\cancel{= \varphi(x-at) + \int_0^t \left(\int_{-\infty}^{+\infty} \delta(\xi-at) f(\tau,x-\xi) dx\right) d\tau = \varphi(x-at) + \int_0^t}$

$$= \delta(x-at) * \varphi(x) + \int_0^t \delta(x-a(t-\tau)) * f(\tau,x)\,d\tau$$

$$= \int_{-\infty}^{+\infty} \delta(\xi-at)\varphi(x-\xi)\,d\xi + \int_0^t \left[\int_{-\infty}^{+\infty} \delta(\xi-at+a\tau)\cdot f(\tau,x-\xi)\,d\xi\right) d\tau$$

$$= \varphi(x-at) + \int_0^t f(\tau, x-a(t-\tau))\,d\tau$$

复习.

平面极坐标分离变量

对固有函数展开.

"非齐次定解问题的分离变量

有界区域，非齐次方程，齐次边界问题.　　注意：找特解要满足齐次边界.

方法：找该特解时，可先找出满足非齐次方程的特解，

再根据齐次边界条件~~拔拔~~分析，最终找到满足的特解

贝塞尔方程的固有值问题

勒让德固有值问题.

贝塞尔方程中 $\rho=x$，而勒让德里 $\rho=1$. 求积分时要注意

写出定解问题

基本解问题中的变量代换、基本解求解初值问题.