§. 平均场理论 (Meanfield Theory)

1. 思路

↑ ↓ ↑ ↓ ↓
↓ ↑ ↑ ↓ ↑
↓ ↓ ↑ ↑ ↑

假设 $\langle S_i \rangle = m$.　　忽略周围的涨落.

单独研究一个自旋，受到外场与周围自旋的平均场.

$$E_1 = -\mu H S_1 - zJmS_1$$ m是平均的结果，是参数.

$$= -\mu H_{eff} S_1 \Rightarrow H_{eff} = H + \frac{zJm}{\mu}$$

2. 平均场方程.

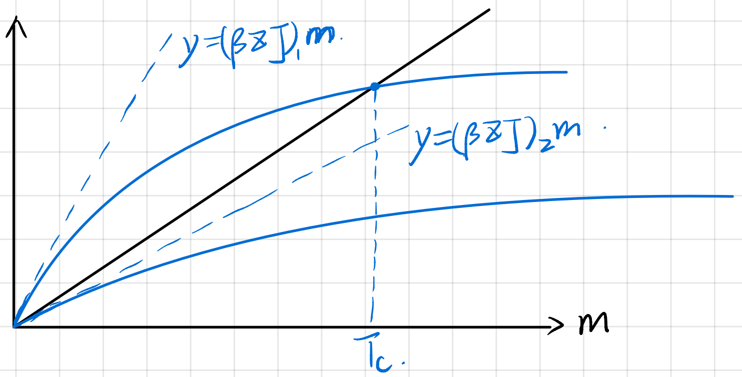
$$\langle S_1 \rangle = \frac{1}{g} \sum_{S_1} S_1 e^{-\beta E_1} \qquad g = \sum_{S_1} e^{-\beta E_1} = e^{\beta\mu H_{eff}} + e^{-\beta\mu H_{eff}}$$

$$\langle S_1 \rangle = \tanh(\beta\mu H + \beta zJm) \left(\frac{\partial \ln g}{\partial(\beta\mu H)}\right) \qquad = 2\cosh(\beta\mu H + \beta zJm)$$

而 $\langle S_1 \rangle = m$.　　注：此时 $Q = g^N$ 而非 $\frac{g^N}{N!}$，因为格点是定域的

得到方程　$m = \tanh[\beta(\mu H + zJm)]$

3. $H=0$ 情况.　$m = \tanh(\beta zJm)$.　$m \to 0$ 时 $\tanh(\beta zJm) \sim \beta zJm$



即出现 T_c 要求 $\beta zJ > 1$.

临界 $T_c \Rightarrow \beta_c zJ = 1 \Rightarrow T_c = \frac{zJ}{k_B}$.

$$\begin{cases} T < T_c, & \beta zJ > 1, \ m \neq 0 \\ T > T_c, & \beta zJ < 1, \ m = 0 \end{cases}$$

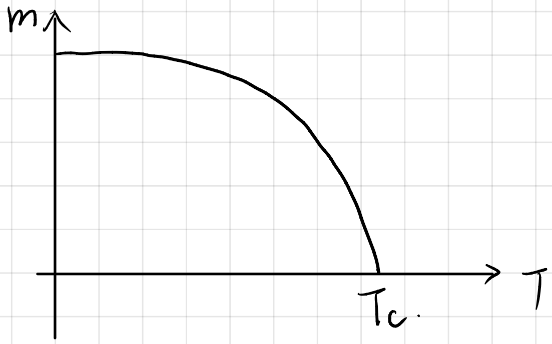
$d=1 \quad T_c^{MFT} = \frac{2J}{k_B}$

$d=2 \quad T_c^{MFT} = \frac{4J}{k_B}$

$\Rightarrow$ 高估 T_c. (因为平均场忽略了周围环境的涨落，相当于认为周围粒子已经有序因此高估了)

但事实上当维度升高时，平均场是越来越精准. $d \geq 4$ 之后很有效.

临界指数.（Critical Exponent）



$m \sim |T-T_c|^{\beta}$

$x\to 0$ 时, $\tanh x \sim x - \frac{x^3}{3}$

$\beta z J = \frac{zJ}{kT}$ 而 $zJ = kT_c \Rightarrow \beta z J = \frac{T_c}{T}$

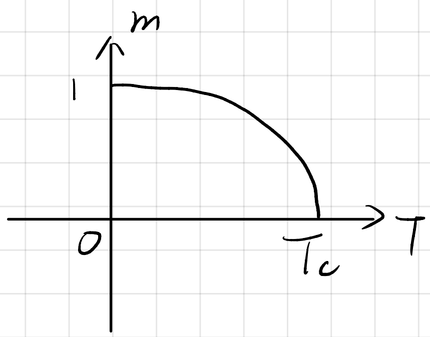
则 $m = \left(\frac{T_c}{T}\right)m - \left(\frac{T_c}{T}\right)^3 \frac{m^3}{3}$

$\Rightarrow m^3 = 3\left(\frac{T}{T_c}\right)^3\left(\frac{T_c - T}{T}\right)m$ （$T\to T_c$, $m \ll 1$）

定义 $\tau = \frac{T-T_c}{T_c}$ 则 $m\tau + \frac{1}{3}\left(\frac{T_c}{T}\right)^2 m^3 = 0$

$\Rightarrow T\to T_c^+$, $m=0$

$T\to T_c^-$, $m \sim \pm\sqrt{\frac{3}{T_c}}(T_c - T)^{\frac{1}{2}}$ 即 $\beta = \frac{1}{2}$



依据 β 的不同，分为不同的相变分类，类别与对称性有关！

4. $H\neq 0$ 的情况.（$H\to 0^+$, $T\to T_c$）

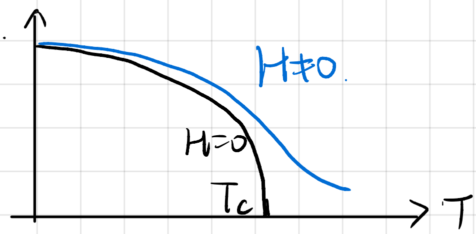
$m = \tanh\left[\beta\mu H + \frac{T_c}{T}m\right]$ 利用 $\tanh x \simeq x - \frac{1}{3}x^3$.

$\simeq \left(\beta\mu H + \frac{T_c}{T}m\right) - \frac{1}{3}\underbrace{\left(\frac{T_c}{T}\right)^3}_{\text{认为是1}} m^3$ （认为 $H^2 \sim o(m^3)$）

$\Rightarrow \boxed{\tau m + \frac{1}{3}m^3 \simeq \beta_c \mu H}$

注：之前的推导中 $m\tau + \frac{1}{3}m^3 = 0$, 即现在外场带来一个小量 $\beta_c \mu H$.

(i) $H\neq 0$ 总有非零解.

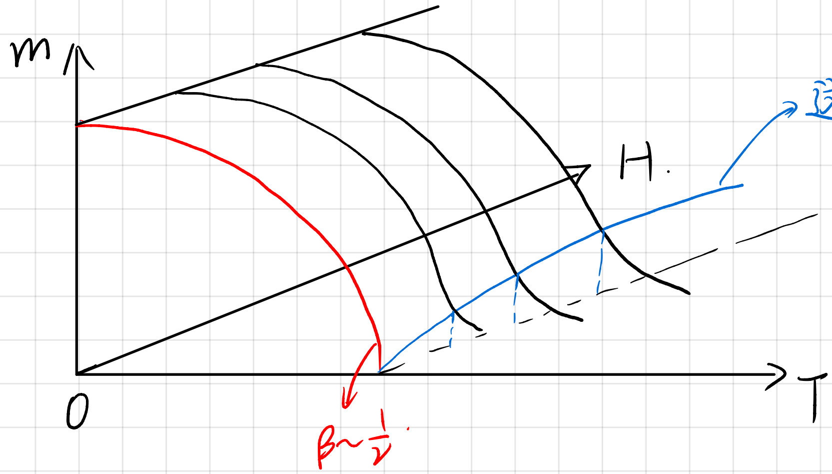


(ii) pV系统的等温线. $pV = const$ (idea gas).

H, M系统, $T=T_c$ 时, $\tau=0$.

由之前的等式给出, $\frac{\mu H}{k_B T_c} = \frac{1}{3} m^3 \longrightarrow H \sim m^3$. $\delta = 3$.

即 $H m^{-3} = const$.



5. $\chi = \frac{1}{N}\left(\frac{\partial M}{\partial(\beta H)}\right)_\beta = \frac{kT}{N}\left(\frac{\partial M}{\partial(\beta H)}\right)_\beta = \mu k T\left(\frac{\partial m}{\partial H}\right)$ $(M = Nm\mu)$

利用 $\tau m + \frac{1}{3} m^3 = \beta_c \mu H$, 两边同时对H求导

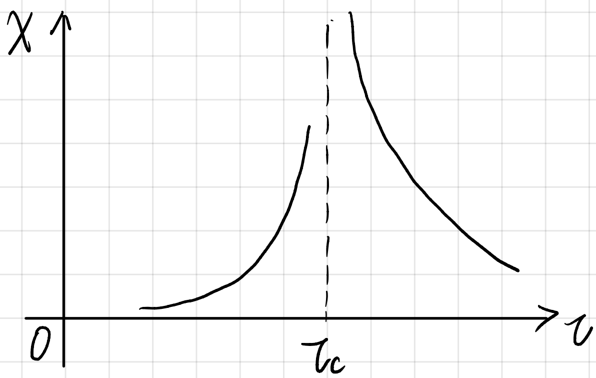
$$\tau\left(\frac{\partial m}{\partial H}\right) + m^2\left(\frac{\partial m}{\partial H}\right) = \beta_c \mu.$$

$$\Rightarrow (m^2+\tau)\left(\frac{\partial m}{\partial H}\right) = \beta_c \mu.$$

则 $\chi = \mu k_B T \beta_c \mu / (m^2+\tau) = \left(\frac{T}{T_c}\right)\frac{\mu^2}{m^2+\tau}$ $T\to T_c, \frac{T}{T_c}=1$ $(\chi \sim \langle \delta M^2 \rangle)$

$$\chi = \begin{cases} \sim \mu^2/\tau & \tau>0,\ m=0 \\ \sim \mu^2/(-2\tau) & \tau<0,\ m^2=-3\tau \end{cases}$$ (利用H=0的结论).

即 $\chi \sim |\tau|^{-1}$ 发散! $\gamma = 1$.



一些标度律　　$C \sim |\tau|^{-\alpha}$.　　$m \sim (-\tau)^{\beta}$ ($T > T_c$ 时 $m=0$)

$\chi \sim |\tau|^{-\gamma}$　　$h \sim m^{\delta}$.

$\xi \sim |\tau|^{-\nu}$ (关联长度).

从平均场中我们得到 $\alpha=0$. $\beta=\frac{1}{2}$　$\gamma=1$. $\delta=3$.

有一些标度律: $\alpha+2\beta+\gamma=2$.

6. 能量

$$\langle E_i \rangle_{mf} = -\mu H m - zJm^2$$

但这部分明显算重复了!

$$\langle E_{tot} \rangle_{mf} = N \langle E_i \rangle_{mf} = -N(\mu H m + \underline{zJm^2})$$

事实上 $\langle E_{tot} \rangle \simeq \langle E_{tot} \rangle_{mf} + \frac{1}{2} N z J m^2$

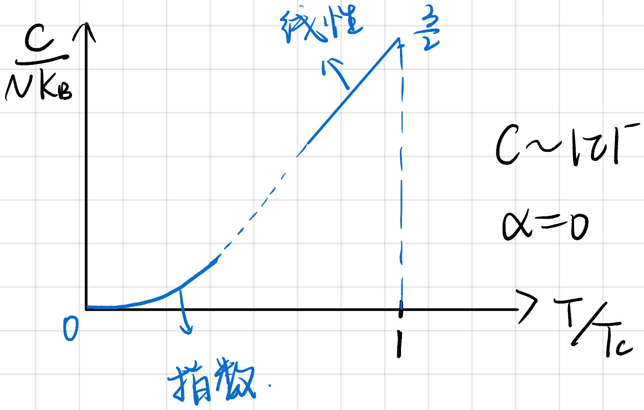
$$= -N\mu H m - \frac{1}{2} N z J m^2 \triangleq \langle E \rangle$$

$$Q \simeq Q_{mf} \cdot \underbrace{e^{-\beta(\frac{1}{2}NzJm^2)}}_{\text{能量校正因子}}$$

$$Q \simeq e^{-\frac{1}{2}NzJm^2}\{2\cosh[(\beta\mu H + \beta zJm)]\}^N.$$

7. $C = \frac{\partial E}{\partial T}\Big|_{H=0} = -\frac{1}{2}NzJ\frac{\partial}{\partial T}(m^2)$

$$= \begin{cases} 0, & \tau > 0 \quad (m=0) \\ \frac{3}{2}Nk_B. & \tau \to 0^- \quad (\text{此时 } m^2 = -3\tau,\ \frac{\partial m^2}{\partial T} = -\frac{3}{T_c} \\ & \text{且 } zJ = k_B T_c) \end{cases}$$



$C \sim |\tau|^{-\alpha}$.

$\alpha = 0$

(* 思路. $m = \tanh(\frac{T_c}{T}m)$.

两边对 T 求导.

$$\frac{dm}{dT} = \frac{1}{\cosh^2(\frac{T_c}{T}m)}\left(\frac{T_c}{T}\frac{dm}{dT} - \frac{T_c}{T^2}m\right)$$

$$\Rightarrow \frac{dm}{dT} = -\frac{T_c}{T^2}m \Big/ \left(\cosh^2(\frac{T_c}{T}m) - \frac{T_c}{T}\right)$$

可以得到 $C=Nkx^2/\{\cosh^2(x)-\frac{x}{\tanh(x)}\}$ $(x=\frac{T_c}{T}m)$)

$T\ll T_c,\ x\to\infty\ (m\to 1)$ $C\simeq 4Nkx^2e^{-2x}$.

8. 自由能 $G(T,H)=-kT\ln Q$.

$$=\frac{1}{2}NzJm^2-NkT\ln(2\cosh[\beta\mu H+\beta zJm])$$

利用 $\frac{1}{\cosh^2 x}=1-\tanh^2 x$. 而 $\tanh(\beta\mu H+\beta zJm)=m$

可得 $G(T,H)=\frac{1}{2}NzJm^2-NkT\ln\left[2\left(\frac{1}{1-m^2}\right)^{\frac{1}{2}}\right]$.

$$=\frac{1}{2}NzJm^2+\frac{1}{2}NkT\ln(1-m^2)-NkT\ln 2.$$

$m\to 0$ 时, $\ln(1-m^2)\sim -m^2+\frac{1}{2}m^4$

则 $\lim_{m\to 0}G(T,H)\simeq N\underbrace{\left(-\frac{1}{2}k_BT_c\tau m^2-\frac{1}{4}k_BT_cm^4-k_BT_c\ln 2\right)}_{g(T,H)}$

H隐含在m中.

求 $\frac{1}{N}F(T,m)=\frac{1}{N}G(T,H)+H\frac{M}{N}$ → μm.

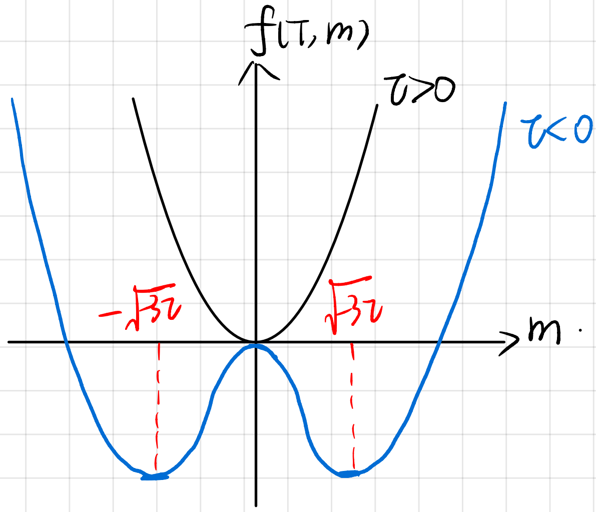
$f(T,m)$

利用G-W方程: $mH=(m\tau+\frac{1}{3}m^3)k_BT_c\cdot m$

$$=\frac{1}{2}k_BT_c\tau m^2+\frac{1}{12}k_BT_cm^4-k_BT\ln 2.$$

令 $\frac{\partial f(T,m)}{\partial m}=0 \Rightarrow (\tau m+\frac{1}{3}m^3)=0$

恰好是H取零时平衡态m满足的方程.



* laudau 自由能(唯象)

$F(m,T)\sim a(T)+\frac{b(T)}{2}m^2+\frac{1}{4}c(T)m^4+\ldots$

我们之前通过对微观态的分析一步步得到配分函数等,而laudau通过对称性出发,从现象分析相变的理论.

Laudau-Ginzburg. $m=m(\vec{r})$ 局域有平均场而宏观上有起伏~泛函.

8. 重整化群方法初步

1. 基本思想：

(1) 粗粒化

以 1D Ising 为例. $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ $\boxed{5}$ $\boxed{6}$ … (N 个自旋相互作用 $\underbrace{\beta J}_{K}$).

↓ 若只保留奇数自旋.

$\boxed{1}$ $\boxed{3}$ $\boxed{5}$ … ($\frac{N}{2}$ 个自旋相互作用 K')

若能找到 K, K' 的关系, 则此过程可以一直重复, 直到最基本的结构. (利用临界点附近标度不变)

① $K \to K' \to K'' \to \cdots$

② $Q(N,K) \to Q(\frac{N}{2},K') \to Q(\frac{N}{4},K'') \to \cdots$

③ 一直重复直到不动点, (代表临界点)

2. 1D Ising 链 $(H=0)$.

(1) $E_\nu = -J(S_1S_2 + S_2S_3 + \cdots + S_{N-1}S_N + S_NS_1)$

$$Q(K,N) = \sum_{\{S_i=\pm1\}} e^{K(S_1S_2+S_2S_3+\cdots+S_NS_1)} \quad (\beta J = K)$$

$$= \sum_{\{S_1,S_3,S_5\cdots\}} \underbrace{\left(\sum_{S_2} e^{K(S_1S_2+S_2S_3)} \sum_{S_4} e^{K(S_3S_4+S_4S_5)} \cdots \right)}$$

若括号内可以写为 $e^{K'(S_1S_3+S_3S_5+\cdots)}[f(K)]^{\frac{N}{2}}$

则 $Q(N,K) = [f(K)]^{\frac{N}{2}} Q(\frac{N}{2},K')$.

(2) $\sum_{S_2} e^{K(S_1S_2+S_2S_3)} = e^{K'S_1S_3}$ 对任意 S_1, S_3 均成立!

$$\Rightarrow e^{K(S_1+S_3)} + e^{-K(S_1+S_3)} = f(K)\, e^{K'S_1S_3}$$

考察 $e^{K(S+S')} + e^{-K(S+S')} = f(K)\, e^{K'SS'}$ 对任意 $S\,S'$ 均成立!

$S+S'$

$\left.\begin{array}{ll} 2 & (S=S'=1) \\ -2 & (S=S'=-1) \end{array}\right\} \Rightarrow \underset{\sim\sim\sim}{e^{2K}+e^{-2K}=f(K)e^{K'}}$ ①

$0 \left\langle\begin{array}{l} S=1 \ S'=-1 \\ S=-1 \ S'=1 \end{array}\right\} \Rightarrow \underset{\sim\sim\sim}{2=f(K)e^{-K'}}$ ②

利用①.②两式 $\Rightarrow \begin{cases} f(K)=2\sqrt{\cosh(2K)} \\ K'=\frac{1}{2}\ln[\cosh(2K)] \end{cases}$

注意到：$\cosh(2K)=\frac{1}{2}(e^{2K}+e^{-2K})\leq e^{2K}\quad (K\geq 0)$

$K'\leq\frac{1}{2}\ln e^{2K}=K$. 即粗粒化中相互作用减弱.

定义 $g(K)=\frac{1}{N}\ln Q(N,K)$.

$=\frac{1}{N}\ln\{[f(K)]^{\frac{N}{2}}Q(\frac{N}{2},K')\}$

$=\frac{1}{2}\ln[f(K)]+\underbrace{\frac{1}{N}\ln Q(\frac{N}{2},K')}_{\sim\sim\sim}\cdot\frac{1}{2}g(K')$ $\quad\left(g(K')=\frac{1}{\frac{N}{2}}\ln Q(\frac{N}{2},K')\right)$

$\Rightarrow g(K)=\frac{1}{2}g(K')+\frac{1}{2}\ln f(K)$

$\Rightarrow g(K')=2g(K)-\ln f(K)$.

粗化：

(a) $K'=\frac{1}{2}\ln[\cosh(2K)]$

(b) $g(K')=2g(K)-\ln f(K)$

$\Downarrow$

粗化过程中 K会减小到 0

$g(0)=\frac{1}{N}\ln\underbrace{Q(N,0)}_{\hookrightarrow 2^N}=\ln 2$.

细化：

(c) $K=\frac{1}{2}\operatorname{arccosh}(e^{2K'})$

(d) $g(K)=\frac{1}{2}g(K')+\frac{1}{2}K'+\frac{1}{2}\ln 2$.

(利用 $K'=\ln\frac{f(K)}{2}$)

细化过程中 K会增大到无穷.

$g(K\to\infty)=\frac{1}{N}\ln\underbrace{Q(N,\infty)}_{\hookrightarrow 2e^{NK}}\simeq K$.

3. 1D Ising链 (H≠0). [N个自旋$(K,h) \to \frac{N}{2}$个自旋(K',h')]

① $Q(N,K)=\sum_{S_1,S_2\ldots S_N} e^{K(S_1S_2+S_2S_3+\cdots)+\underbrace{h(S_1+S_2+S_3+\cdots)}}$ $h=\beta\mu H$

$\to \frac{h}{2}(S_1+S_3)+hS_2+\frac{h}{2}(S_3+S_5)+S_4+\cdots$

$$=\sum_{S_1,S_3\ldots}\left(\sum_{S_2} e^{K(S_1S_2+S_2S_3)+\frac{h}{2}(S_1+S_3)+hS_2}\sum_{S_4} e^{K\cdots}\right).$$

② 要求 $\sum_{S_2} e^{K(S_1S_2+S_2S_3)+\frac{h}{2}(S_1+S_3)+hS_2} = f(K,h)\,e^{K'(S_1S_3)+\frac{h'}{2}(S_1+S_3)}$.

$$\Rightarrow 2\cosh[K(S_1+S_3)+h]\,e^{\frac{h}{2}(S_1+S_3)} \;//$$

(i) $S_1=S_3=1$ $\quad 2\cosh(2K+h)\,e^{h}=f(K,h)\,e^{K'+h'}$

(ii) $S_1=S_3=-1$ $\quad 2\cosh(-2K+h)\,e^{-h}=f(K,h)\,e^{K'-h'}$

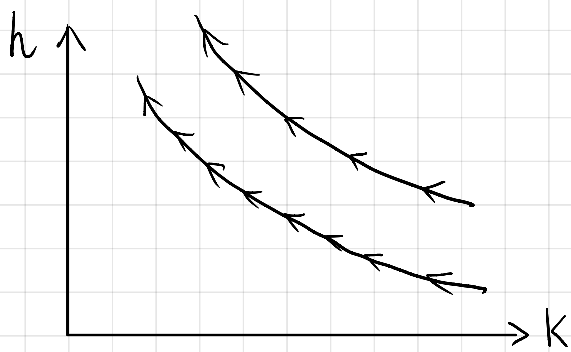
(iii) $S_1=-S_3=1$ $\quad 2\cosh(h) = f(K,h)\,e^{-K'}$

$$\Rightarrow h'=h+\frac{1}{2}\ln\frac{\cosh(2K+h)}{\cosh(2K-h)} \qquad K'=\frac{1}{4}\ln\frac{\cosh(2K+h)\cosh(2K-h)}{\cosh^2(h)}$$

$$f(K,h)=\frac{2\cosh(h)}{e^{K'}}$$

注意到 $h'>h$. $K'<K$. 即等效场变强,等效相互作用减弱.

粗化过程:



令 $g(K,h)=\frac{1}{N}\ln Q(K,h)$.

$\Rightarrow g(K',h')=2g(K,h)-\ln f(K,h)$.

*估计 $g(1,1)$大小.

无相互作用,有外场.

首先考虑 $g(0,h)=\frac{1}{N}\ln Q(N,0,h)$.

$$Q=\left(\sum_{S_i} e^{\beta\mu H S_i}\right)^N=[2\cosh(h)]^N$$

$$g(0,h)=\ln[2\cosh(h)]$$ (5.22题).

可以从 $g(1,1)$ 一步步粗化到 $g(K\approx 0,h)$, 然后通过逆过程求 $g(1,1)$